Intersecting free subgroups in free amalgamated products of groups

Alexander Zakharov

Moscow State University

July 12, 2011

Intersecting subgroups in free groups

The reduced rank of a free group: $\bar{r}(H) = max\{0, r(H) - 1\}$

Theorem (Hanna Neumann, 1957)

Suppose G is a free group, H_1 and H_2 are finitely generated subgroups in G. Then $H_1 \cap H_2$ is also finitely generated (Howson) and

$$\bar{r}(H_1 \cap H_2) \leqslant 2\,\bar{r}(H_1)\,\bar{r}(H_2)$$

Intersecting subgroups in free products

- Factor-free subgroup of a free product: one that intersects trivially with the conjugates to the factors of the free product.
- Factor-free subgroups are free (Kurosh subgroup theorem).

Theorem (W.Dicks and S.V.Ivanov, 2008)

Suppose $G = G_1 * G_2$, and H_1, H_2 are factor-free subgroups of G with finite ranks. Then $H_1 \cap H_2$ also has finite rank and

$$\overline{r}(H_1 \cap H_2) \leqslant 2\frac{q}{q-2}\overline{r}(H_1)\overline{r}(H_2),$$

q is the minimum of orders > 2 of subgroups of groups G_1, G_2 ($\frac{q}{q-2} = 1$ if $q = \infty$.)
In addition, this is the best possible estimate when G contains an involution and $G \not\cong \mathbb{Z}_2 * \mathbb{Z}_2$.

Intersecting subgroups in amalgamated free products

Theorem (A.Z.)

Suppose $G = G_1 *_T G_2$, T is normal finite amalgamated subgroup, and H_1, H_2 are factor-free subgroups of G with finite ranks. Then the intersection $H_1 \cap H_2$ also has finite rank, and

$$\overline{r}(H_1 \cap H_2) \leqslant 2\frac{q^f}{q^f - 2}|T| \cdot \overline{r}(H_1)\overline{r}(H_2),$$

where q^f is the minimum of orders > 2 of subgroups of groups G_1/T, G_2/T. ($\frac{q^f}{q^f-2} = 1$ if $q^f = \infty$).
In addition, this is the best possible estimate when G_1/T or G_2/T contains an involution and $G_1/T * G_2/T \ncong \mathbb{Z}_2 * \mathbb{Z}_2$.

Proof of the estimate

- Consider a factorization $\varphi : G_1 *_T G_2 \to G_1/T * G_2/T$.
- Let
  $\varphi(H_1) = H_1^f, \quad \varphi(H_2) = H_2^f, \quad \varphi(H_1 \cap H_2) = L \subseteq H_1^f \cap H_2^f$
- Since H_1 and H_2 are factor-free,
  $$H_1 \cong H_1^f, \quad H_2 \cong H_2^f, \quad H_1 \cap H_2 \cong L.$$
- Dicks-Ivanov theorem:
  $$\overline{r}(H_1^f \cap H_2^f) \leq 2\frac{q^f}{q^f - 2}\overline{r}(H_1^f)\overline{r}(H_2^f)$$
- L is a subgroup in a free group $H_1^f \cap H_2^f$, so, according to Schreier formula,
  $$\overline{r}(L) = |H_1^f \cap H_2^f : L|\, \overline{r}(H_1^f \cap H_2^f)$$

Proof of the estimate

- We obtain

$$\overline{r}(H_1 \cap H_2) \leq 2\frac{q^f}{q^f - 2}|H_1^f \cap H_2^f : L|\,\overline{r}(H_1)\overline{r}(H_2)$$

- Finally,

$$|H_1^f \cap H_2^f : L| \leq |T|,$$

since all the right cosets of L in $H_1^f \cap H_2^f$ have the form

$$\varphi(H_1 \cap H_2 t), \quad t \in T : \quad H_1 \cap H_2 t \neq \varnothing$$

This proves the estimate. □

Notice that the last inequality turns into equality when $T \subseteq H_1 H_2$.

Graph of a subgroup in a free product

To prove that the estimate is the best possible we will use the same graph-theoretic approach as S.V.Ivanov.

Suppose H is a factor-free subgroup of $G = \prod^* G_\alpha$.
Graph $\Psi^*(H)$ associated with subgroup H:

- 2 types of vertices of $\Psi^*(H)$:

1. Primary vertices: right cosets of H in G;
2. Secondary vertices corresponding to the factor G_α: eqivalence classes of right cosets of H in G, $Hg_1 \sim Hg_2$ if $\exists c \in G_\alpha : Hg_1 c = Hg_2$

- Edges of $\Psi^*(H)$: each primary vertex Hg is connected by one edge to the secondary vertex $[Hg]_\alpha$ for all α.

Labeling the graph

- Assume that all edges of $\Psi^*(H)$ are oriented.
- Assign a label to every edge e of $\Psi^*(H)$ label $\varphi(e)$: if e_1, e_2 are edges beginning in primary vertices Hg_1, Hg_2 and ending in the same secondary vertex $[Hg_1]_\alpha = [Hg_2]_\alpha$, then

1. $\varphi(e_i) \in G_\alpha$,
2. $\varphi(e_i^{-1}) = \varphi(e_i)^{-1}$ and
3. $Hg_1\varphi(e_1)\varphi(e_2)^{-1} = Hg_2$

(picture!)

- Base vertex: primary vertex H.
- $\Psi(H)$: minimal connected subgraph of $\Psi^*(H)$, containing all reduced cycles and the base vertex.

Facts about the graph

- Let w be a nonempty reduced word in the alphabet $\bigcup G_\alpha$.

$w \in H$ if and only if there exists a reduced closed path in $\Psi(H)$ ending in the base vertex labeled with w.

- T — maximal subtree in $\Psi(H)$, C — set of edges of $\Psi(H)$, not belonging to T. For every $e \in C$ find path q_e in T from the base vertex to the beginning of e and path r_e in T from the end of e to the base vertex.
  Then H is freely generated by
$$\varphi(q_e e r_e), e \in C.$$
- H has finite rank if and only if $\Psi(H)$ is finite.
- $\bar{r}(H) = -\chi(\Psi(H))$

Cases to be considered

$$\varphi : G = G_1 *_T G_2 \to G_1/T * G_2/T = G_1^f * G_2^f = G^f.$$

q^f is the minimum of orders > 2 of subgroups of G_1^f, G_2^f
Suppose G^f contains an involution and $G^f \not\cong \mathbb{Z}_2 * \mathbb{Z}_2$.

q^f is prime > 2, or $q^f = 4$, or $q^f = \infty$, and G^f has one of the following subgroups:

1. $\mathbb{Z}_2 * \mathbb{Z}_p$, if $q^f = p$, p is prime, $p > 2$
2. (a) $\mathbb{Z}_2 * \mathbb{Z}_4$ or (b) $\mathbb{Z}_2 * (\mathbb{Z}_2 \times \mathbb{Z}_2)$, if $q^f = 4$.
3. $\mathbb{Z}_2 * \mathbb{Z}_2 * \mathbb{Z}_2$, if $q^f = \infty$

It is enough to prove that our estimate is the best possible in the case when G^f is one of the groups above.

When there is equality in our estimate

We need to construct examples of subgroups such that

$$\bar{r}(H_1 \cap H_2) = 2\frac{q^f}{q^f - 2}|T| \cdot \bar{r}(H_1)\bar{r}(H_2)$$

It holds when (remind $H_1^f = \varphi(H_1)$, $H_2^f = \varphi(H_2)$)

1. $$\bar{r}(H_1^f \cap H_2^f) = 2\frac{q^f}{q^f - 2}\bar{r}(H_1^f)\bar{r}(H_2^f)$$

   (equality in Dicks-Ivanov estimate)
2. $T \subseteq H_1 H_2$

Below we give some conditions when 1 holds.

When there is equality in Dicks-Ivanov estimate

Lemma (Dicks, Ivanov)

Suppose H_1^f and H_2^f are factor-free subgroups of finite index in G^f, and G^f is one of the groups mentioned above (1, 2a, 2b, 3). Then Dicks-Ivanov estimate turns into equality if and only if

$$|G^f : H_1^f \cap H_2^f| = |G^f : H_1^f| \cdot |G^f : H_2^f|$$

In particular, if $H_1^f \lhd G^f$ and $H_1^f H_2^f = G^f$, then Dicks-Ivanov estimate turns into equality.

Idea of the proof

To prove that the estimate is the best possible we do the following:

- for every $n = |T|$ for each of the groups G^f mentioned above (1, 2a, 2b, 3) construct factor-free subgroups H_1^f and H_2^f with finite index in G^f, such that $H_1^f \lhd G^f$ and $H_1^f H_2^f = G^f$;
- choose the inverse images of the free generators of constructed subgroups under the homomorphism φ so that $T \subseteq H_1 H_2$.

Case 1. $G^f = \mathbb{Z}_2 * \mathbb{Z}_p$

Suppose $G^f = \langle a, b \mid a^p = b^2 = 1 \rangle$, where p is prime, $p > 2$.
$G_0^f = \langle\langle (ab)^6 \rangle\rangle \subseteq G^f$,
$R = G^f / G_0^f \cong \langle a, b \mid a^p = b^2 = (ab)^6 = 1 \rangle = T(p, 2, 6)$.

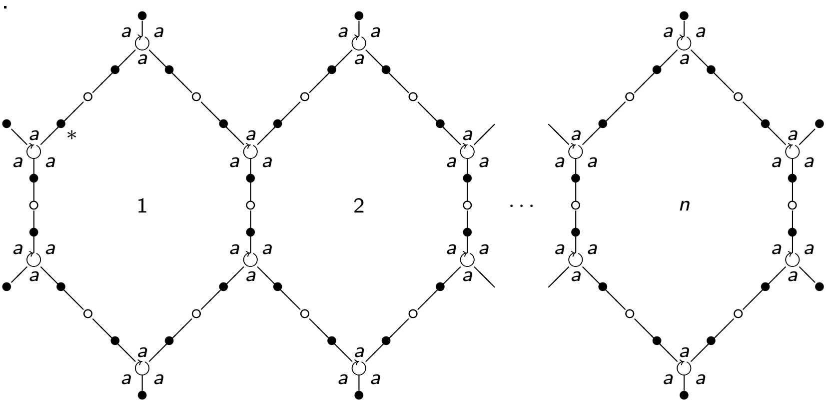
- R is a triangle group
- R is infinite, since

$$\frac{1}{p} + \frac{1}{2} + \frac{1}{6} \leq 1 \quad \text{when} \;\; p \geq 3.$$

- R is residually finite. Therefore, for each finite subset M of group R there exists a homomorphism from R on a finite group, injective on the set M.

Consider the case $p = 3$, the same method can be used for other p.
Consider the following part of $\Psi(G_0^f)$:

Case 1. Part of $\Psi(G_0^f)$ or $\Psi(H_1^f)$



$$w_1^f = (ab)^6,\ w_2^f = ((ab)^6)^{ba^{-1}ba^{-2}}, \ldots, w_n^f = ((ab)^6)^{(ba^{-1}ba^{-2})^{n-1}}$$

— free generators

Case 1. Constructing H_1^f

Take as M the set of the right cosets, corresponding to the primary vertices of this graph.
$\exists \pi : R \to S = R/R_0$:

- π is injective on the set M
- S is a finite group.

Then take $H_1^f : \; S \cong G^f / H_1^f$

- $H_1^f \triangleleft G^f$,
- $G_0^f \subseteq H_1^f$.
- H_1^f has finite index in G^f, since S is finite.
- H_1^f is factor-free
- $\Psi(H_1^f)$ has the same part as $\Psi(G_0^f)$, see picture above.
- Choose $w_1^f, \ldots, w_n^f$ as part of free generators of H_1^f corresponding to this part of $\Psi(H_1^f)$.

Case 1. Constructing H_2^f

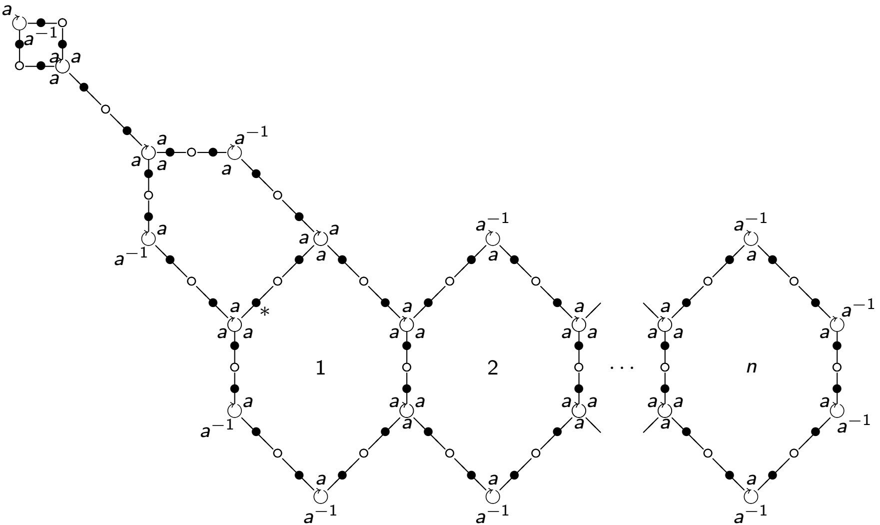
$K \subseteq G^f$, (freely) generated by the following elements:

$$w_1^f, \dots, w_n^f, \quad (ba)^5, \quad (ba)^2(ba^{-1})^5(ba)^2.$$

The graph $\Psi(K)$ — see picture!

- K is a factor-free subgroup of G^f
- K has infinite index in G^f.
- Construct a subgroup $H_2^f \subseteq G^f$:
  $K \subseteq H_2^f$, H_2^f is factor-free in G^f and H_2^f has finite index in G^f
- Notice that $w_1^f, \dots, w_n^f$ belong to free generators of both subgroups H_1^f, H_2^f.

Case 1. $\Psi(K)$, also part of $\Psi(H_2^f)$



Case 1. Proving that $H_1^f H_2^f = G^f$

$$(ba)^6 \in H_1^f,\ (ba)^5 \in H_2^f \Rightarrow ba \in H_1^f H_2^f;$$

$$(ba)^2(ba^{-1})^5(ba)^2 \in H_2^f \Rightarrow (ba^{-1})^5 \in H_1^f H_2^f;$$

$$(ba^{-1})^6 = ((ab)^6)^{-1} \in H_1^f \Rightarrow ba^{-1} \in H_1^f H_2^f;$$

$\Rightarrow a^2 \in H_1^f H_2^f$, but $a^p = 1$, p is odd

$\Rightarrow a, b \in H_1^f H_2^f \Rightarrow H_1^f H_2^f = G^f$.

Case 1. Choosing the inverse images

Now choose the inverse images of free generators of subgroups H_1^f and H_2^f under homomorphism φ as following.

- Let $T = \{t_1, ..., t_n\}$.
- Choose arbitrary inverse images $w_1, ..., w_n \in G$ of $w_1^f, ..., w_n^f$ considered as the generators of H_1^f.
- Choose inverse images $w_1 t_1, ..., w_n t_n$ of the same elements $w_1^f, ..., w_n^f$ considered as the generators of H_2^f.
- Choose arbitrary inverse images of all other free generators of H_1^f and H_2^f.
- Call H_1 and H_2 the obtained inverse images of groups H_1^f and H_2^f respectively.
- $T \subseteq H_1 H_2$ holds.

So in this case our estimate is the best possible.

Case 2a. $G^f = \mathbb{Z}_2 * \mathbb{Z}_4$

Suppose $G^f = \langle a, b \mid a^4 = b^2 = 1 \rangle$.

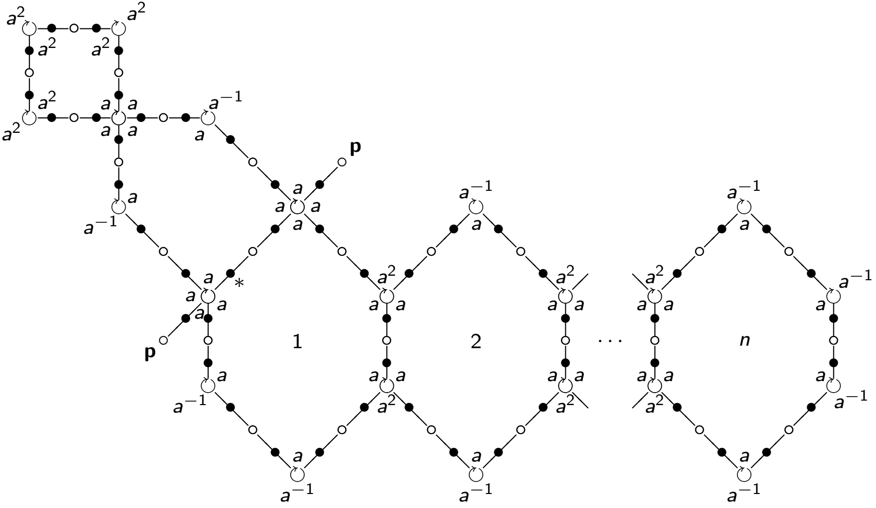
- H_1^f is constructed in the same way as in the previous case.
- $K \subseteq G^f$, (freely) generated by the following elements:

  $$w_1^f, \dots, w_n^f, \quad (ba)^5, \quad (ba)^2(ba^2)^5(ba)^2, \quad (ba^2)^2$$

  The graph $\Psi(K)$ — see picture!
- As in the previous case, K is factor-free in G^f, but K has an infinite index in G^f.
- Again construct $H_2^f \subseteq G^f$:
  $K \subseteq H_2^f$, H_2^f is factor-free in G^f and H_2^f has finite index in G^f
- Again show that $H_1^f H_2^f = G^f$.
- Choose the inverse images as in the previous case.

So in this case our estimate is the best possible.

Case 2a. $\Psi(K)$, also part of $\Psi(H_2^f)$



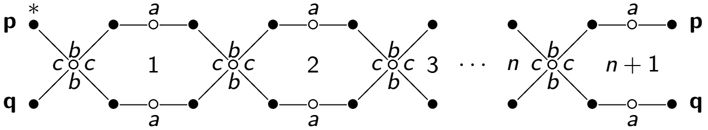
Case 2b. $G^f = \mathbb{Z}_2 * (\mathbb{Z}_2 \times \mathbb{Z}_2)$. Constructing H_1^f

Suppose $G^f = \langle a, b, c \mid a^2 = b^2 = c^2 = bcbc = 1 \rangle$

Take $H_1^f = \langle\langle acac, \quad (ab)^{n+1} \rangle\rangle$, then

$G^f / H_1^f \cong D_{n+1} \times \mathbb{Z}_2$, where D_n is dihedral group of order $2n$.

The graph $\Psi(H_1^f)$:



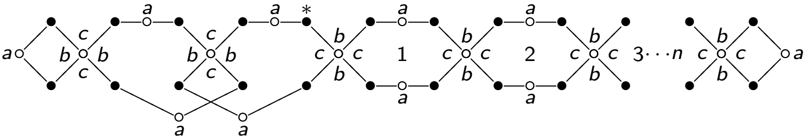
- H_1^f is factor-free
- H_1^f has finite index in G^f
- $w_1^f = (acac)^b$, $w_2^f = (acac)^{bab}$, ..., $w_n^f = (acac)^{(ba)^{n-1}b}$ — belong to free generators of H_1^f.

Case 2b. Constructing H_2^f

$H_2^f \subseteq G^f$, (freely) generated by the following elements:

$$w_1^f, \ldots, w_n^f, \quad (ab)^{acac},\ acababa,\ abcac,\ (ac)^{(ba)^n b}$$

The graph $\Psi(H_2^f)$:



- H_2^f is factor-free
- H_2^f has finite index in G^f

Case 2b: last steps

- Show that $H_1^f H_2^f = G^f$. Indeed,

$$acac \in H_1^f,\ (ab)^{acac} \in H_2^f \Rightarrow ab \in H_1^f H_2^f.$$

$$abcac,\ ab \in H_1^f H_2^f \Rightarrow cac,\ acac \in H_1^f H_2^f \Rightarrow a, b \in H_1^f H_2^f.$$

$$acababa, a, b \in H_1^f H_2^f \Rightarrow c \in H_1^f H_2^f,$$

so $H_1^f H_2^f = G^f$.

- Choose the inverse images of free generators of subgroups H_1^f and H_2^f in the same way, as in the first case (with new $w_1^f, ..., w_n^f$).

So in this case our estimate is the best possible.

Case 3. $G^f = \mathbb{Z}_2 * \mathbb{Z}_2 * \mathbb{Z}_2$

Suppose $G^f = \langle a, b, c \mid a^2 = b^2 = c^2 = 1 \rangle$.

This case can be reduced to the previous one.

Let $G^f_*, H^f_{1*}, H^f_{2*}$ denote groups G^f, H^f_1, H^f_2 from the previous case respectively.
$(G^f_* = \mathbb{Z}_2 * (\mathbb{Z}_2 \times \mathbb{Z}_2))$

Consider a factorization

$$\psi : G^f \to G^f_* \cong G^f / \langle\langle bcbc \rangle\rangle.$$

Case 3, the last one

Let H_1^f, H_2^f denote full inverse images of H_{1*}^f, H_{2*}^f under homomorphism ψ.

- $H_1^f \lhd G^f$;
- H_1^f, H_2^f are factor-free;
- H_1^f, H_2^f have finite index in G^f;
- $H_1^f H_2^f = G^f$.

Choose the inverse images of free generators of subgroups H_1^f and H_2^f in the same way, as in the previous cases.

So our estimate is the best possible. $\square$