

On the SQ-universality of $C(6)$ -groups

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Definition

A countable group G is said to be SQ -universal if every countable group can be embedded in a quotient of G .

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- many amalgamated free products and *HNN*-extensions (Lossov, 1986; Sacerdote and Schupp, 1974; Fine and Tretkoff, 1979)
- many small cancellation groups.



Free subgroups of small cancellation groups

Let G be a group satisfying the small cancellation conditions $C(p)$ and $T(q)$, with $p, q \geq 3$.

If the presentation is finite and $\frac{1}{p} + \frac{1}{q} < \frac{1}{2}$ then G is word-hyperbolic (Gersten and Short, 1990) and so G is SQ-universal if it has no cyclic subgroup of finite index (Ol'shanskii, 1995).



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Now suppose that $\frac{1}{p} + \frac{1}{q} = \frac{1}{2}$. Observe that if G is SQ-universal then G contains a free subgroup of rank 2. It is known exactly when a group satisfying $C(p)$ and $T(q)$, where $\frac{1}{p} + \frac{1}{q} = \frac{1}{2}$, contains such a free subgroup.



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- 3 Edjvet and Howie (1988) have proved the analogous result for groups satisfying $C(3)$ and $T(6)$.



SQ-universality of small cancellation groups

A more general result would be to classify groups which satisfy one of the standard non-metric cancellation hypotheses and are SQ-universal. We list some known results in this direction.



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- 1 In 1989 Howie has shown which finitely related $C(3), T(6)$ -groups not having a so-called *special presentation* are SQ-universal.
- 2 In 2010 Edjvet and Vdovina have completed the study of this class of groups showing that $C(3), T(6)$ -groups with special presentations are not SQ-universal.



SQ-universality of $C(6)$ -groups

We consider finitely related presentations satisfying condition $C(6)$ and prove the following.

Theorem

Let G be a group having a finitely related presentation satisfying the small cancellation condition $C(6)$. Then G is SQ-universal except when G is cyclic or infinite dihedral.



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Remark. A first attempt at proof of the theorem above has been made by Al-Janabi in his PhD Thesis in 1977.

Unfortunately Al-Janabi's proof is incomplete and contains errors. Here we use his *curvature re-distribution* argument to give a complete proof.



Diagrams

Our proof is based on the analysis of suitable *van Kampen diagrams*. A *map* is a finite planar 2-complex. A *diagram* over a group G is an oriented map M together with a labelling function Φ which associates to each edge e in M an element $\Phi(e) \in G$ such that $\Phi(e^{-1}) = \Phi(e)^{-1}$, where e^{-1} is the oppositely oriented edge of e .

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Let $G = \langle X \mid R \rangle$. A *van Kampen diagram* \mathcal{M} over the presentation $\langle X \mid R \rangle$ is a connected, simply connected diagram over $F(X)$ such that:

- 1 if e is an edge of \mathcal{M} then $\Phi(e) \neq 1$;
- 2 if e_1, \dots, e_j is a boundary cycle of a region $\Delta \in \mathcal{M}$ then $\Phi(\Delta) := \Phi(e_1) \cdots \Phi(e_j)$ is reduced without cancellation and is a cyclically reduced conjugate of some $r^{\pm 1}$, where $r \in R$ (i.e. is an element of R^* , the symmetrized closure of R).

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From now on we will refer to van Kampen diagrams simply as diagrams. If $G = \langle X \mid R \rangle$ satisfies a small cancellation condition then any diagram over $\langle X \mid R \rangle$ has certain geometric constraints.



This is important since, as the following result shows, diagrams can be used for studying membership in a normal subgroup of a free group.

Theorem (van Kampen Lemma)

Consider a presentation $\langle X \mid R \rangle$ and let $w \in F(X)$. Then $w \in N = \overline{R}$ if and only if there exists a diagram over $\langle X \mid R \rangle$ whose boundary label is (a cyclic permutation of) w .

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Remark. One can always assume that a diagram \mathcal{M} is *reduced* in the following sense. If Δ_1 and Δ_2 are any two regions in \mathcal{M} such that there exists an edge $e \in \partial\Delta_1 \cup \partial\Delta_2$ and $e\gamma_1$, $e^{-1}\gamma_2$ are boundary cycles of Δ_1 , Δ_2 , respectively, then $\Phi(\gamma_1) \neq \Phi(\gamma_2)^{-1}$.

We turn now to the geometric consequences which the small cancellation hypotheses have on diagrams.

Proposition

Let $G = \langle X \mid R \rangle$ and let \mathcal{M} be a reduced diagram over $\langle X \mid R \rangle$.

- 1 If R satisfies $C(p)$ then for each region $\Delta \in \mathcal{M}$ such that $\partial\Delta \cap \partial\mathcal{M}$ does not contain an edge we have $d(\Delta) \geq p$.
- 2 If R satisfies $T(q)$ then for each interior vertex v of \mathcal{M} we have $d(v) \geq q$.

Small cancellation and embedding theorems

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Theorem

*Any non trivial free product $G * H$ which is not infinite dihedral (that is it is not the product of two copies of the cyclic group of order 2) is SQ-universal.*

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Let $G = \langle A \mid R \rangle$. Recall that a word $u \in F(A)$, the free group on A , is a piece if there exist words r, s in the symmetrized closure R^* of R such that $r \equiv ur'$, $s = us'$ with r' and s' distinct words and condition $C(k)$ means that no element in R^* is a product of fewer than k pieces. Now suppose that R is finite and satisfies $C(6)$. Note that if $|A| - |R| \geq 2$, in particular if A is infinite, then G is SQ-universal (Baumslag and Pride, 1978).



A sketch of the proof

Moreover, we can assume that each generator is a piece. If this is not the case then using Tietze transformations one can obtain a presentation for G which is either a $C(6)$ presentation on fewer generators or is a free product of a $C(6)$ presentation on fewer generators and a non-trivial factor which is a free product of one or more cyclic groups. In this latter case G is SQ -universal except when it is infinite dihedral. The restriction that $|R| < \infty$ is essential and relates to our method of proof. Thus we need to prove the following.



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Proposition

Let $G = \langle A \mid R \rangle$ be a finite presentation where $|A| \geq 2$. Suppose that R satisfies $C(6)$ and that each generator in A is a piece. Then G is SQ -universal.



A sketch of the proof

Let $F(x, y)$ denote the free group on x and y where $x, y \notin A$; let B be a normal subgroup of $F(x, y)$; and let $B_0 \subseteq B$ be the set of non-empty cyclically reduced words in B . Assume B is chosen such that $|u(x, y)| \geq 6$ in the free group $F(x, y)$ for all $u(x, y) \in B_0$. Let $z_1, z_2 \in F(A)$, $s_1 = xz_1$ and $s_2 = yz_2$. Let $C = \{s_1, s_2\}$ and suppose that z_1 and z_2 are not expressible as a product of fewer than six pieces relative to $R \cup C$ (we will always use *relative to S* , for example, to mean *relative to the symmetrized closure of S*).



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Consider the presentation $\widehat{G} = \langle A, x, y \mid R, B_0, C \rangle$. Clearly \widehat{G} is a quotient of G . Let $K = \langle x, y \mid B_0 \rangle$ and consider the natural map $K \rightarrow \widehat{G}$ sending $x \mapsto x$ and $y \mapsto y$.



A sketch of the proof

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If H is a countable group then H embeds in a group like K

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We want to show that G is SQ-universal. In order to do this it suffices to show that we can always choose suitable z_1 and z_2 in such a way that the above natural map is an embedding.

We will argue by contradiction and suppose that the natural map is not an embedding for any choice of z_1 and z_2 , that is, it has a non-trivial kernel. This implies that there exists a (van Kampen) diagram over $R \cup B_0 \cup C$ with boundary label in $F(x, y) \setminus B$. Among all such diagrams pick one, say \mathcal{M} , with least number of regions. This implies that \mathcal{M} is reduced.



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- A₄** z_1 and z_2 are not expressible as a product of fewer than six pieces relative to $R \cup C$;
- A₅** if w is a subword of both an element of R and an element of C , then w is expressible as the product of no more than three pieces relative to R .

Geometric properties of \mathcal{M}

$\mathcal{M} = \mathcal{V} \cup \mathcal{E} \cup \mathcal{F}$ is an oriented labelled map. There is a natural partition of the set of regions $\mathcal{F} = \mathcal{G} \dot{\cup} \mathcal{K} \dot{\cup} \mathcal{C}$, corresponding to labels in R , B_0 and C , respectively. Assumptions **A**₁ to **A**₅ have the following consequences on \mathcal{M} .

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- if Δ is a \mathcal{C} -region then $d(\Delta) \geq 7$;

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- $\mathcal{C} \neq \emptyset$;

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- $\mathcal{C} \neq \emptyset$;
- $\mathcal{G} \neq \emptyset$.

Definition

Let Δ be a \mathcal{G} -region and v an interior vertex with $v \in \partial\Delta$. Define

$$\underline{d}(v, \Delta) := \frac{3 - d(v)}{v^\#} \quad \text{if } |[(\Delta, \mathcal{C}); v]| = 0$$

$$\underline{d}(v, \Delta) := \frac{3 - d(v)}{v^\#} \left(\frac{1}{|[(\Delta, \mathcal{C}); v]|} + \frac{|[\mathcal{C}; v]| + |[\mathcal{K}; v]|}{|[(\mathcal{G}, \mathcal{C}); v]|} \right) \quad \text{otherwise}$$

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Here $v^\#$ is the number of distinct regions at v ; $|[(\Delta, \mathcal{C}); v]|$ is the number of edges at v which Δ shares with \mathcal{C} -regions; $|[\mathcal{C}; v]|$ is the number of distinct \mathcal{C} regions at v and $|[(\mathcal{G}, \mathcal{C}); v]|$ is the number of edges at v shared by a \mathcal{G} -region and a \mathcal{C} -region.

Definition

Let Δ be a \mathcal{C} -region. Let e be an edge occurring in $\partial\Delta$ and $\sigma(e)$ the region (if it exists) sharing e with Δ . If $\sigma(e) \in \mathcal{G}$ define

$$\tilde{d}(e; \Delta) := \frac{\alpha(\sigma(e)) - i(\sigma(e))}{|(\sigma(e), \mathcal{C})|} - 1 + \frac{2}{|(\sigma(e), \mathcal{C})|} \sum_{v \in \partial\sigma(e)} \underline{d}(v, \sigma(e));$$

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Here $\alpha(\sigma(e)) = 6$ if $\sigma(e)$ is an interior region and $\alpha(\sigma(e)) = 4$ if $\sigma(e)$ is a boundary region.

The key result

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Proposition

There exists a \mathcal{C} -region Δ in \mathcal{M} such that

$$\sum_{e \in \partial \Delta} \tilde{d}(e, \Delta) > -6.$$

We point out that this does not depend on the choice of z_1 and z_2 (provided they are chosen in such a way that \mathbf{A}_4 and \mathbf{A}_5 are satisfied). We get a contradiction to this arguing on a case by case basis as follows.



Getting the contradiction

Put

$$z_1 = w \prod_{j=2}^{L+1} (uw^j) \quad \text{and} \quad z_2 = u \prod_{j=L+2}^{2L} (w^j u);$$

where L is a large enough number; moreover u and w are chosen in such a way that the following are satisfied.

Getting the contradiction

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where L is a large enough number; moreover u and w are chosen in such a way that the following are satisfied.

- 1 There is no cancellation forming the products uw and wu ;

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where L is a large enough number; moreover u and w are chosen in such a way that the following are satisfied.

- 1 There is no cancellation forming the products uw and wu ;
- 2 the first (last) letter of u is not the same as the last (first) letter of w ;

Getting the contradiction

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- 1 There is no cancellation forming the products uw and wu ;
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- 3 for all i , u is not a subword of w^i .

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This ensures that a piece relative to $R \cup C$ must be a subword of $w^i u w^{i+1}$ for some i so **A₄** is satisfied. We are left to choose u and w in such a way that **A₅** is satisfied and that we can deduce the desired contradiction.



Getting the contradiction

As said before this is done on case by case basis. Once the choice for u and w has been made we prove that given any \mathcal{C} -region Δ in \mathcal{M} we have

$$\sum_{e \in \partial\Delta} \tilde{d}(e, \Delta) \leq -6$$

and this clearly contradicts the proposition above. The contradiction comes from the assumption that G is not SQ-universal, hence the main theorem follows.

Working modulo the symmetric closure of R , let R_0 be the set of relators in R which are powers of generators in A . For $a, b \in A$ let a^p and b^q be the maximal powers of a and b , respectively, occurring in elements of $R \setminus R_0$. We will distinguish the following five cases.



Getting the contradiction

- ① $|A| \geq 2$ and there exist two distinct generators $a, b \in A$ such that $a^{m'}, b^{n'} \in R$ for some $m', n' > 0$.
- ② $|A| \geq 2$ and there exist $a, b \in A$ such that
 - (i) $\Theta_R(a^p) = \Theta_R(b^q) = 1$;
 - (ii) $a^m \notin R$ for any $m \neq 0$.
- ③ $|A| \geq 3$ and there exist $a, b \in A$ such that
 - (i) $\Theta_R(a^p) = \Theta_R(b^q) = 2$;
 - (ii) the unique relator r_1 involving a^p is not of the form $r_1 \equiv (a^p b^\varepsilon z_1 b^\eta)^{n_1}$ where $n_1 \geq 1$, $\varepsilon, \eta = \pm 1$ and z_1 is a^p -free;
 - (iii) the unique relator r_2 involving b^q is not of the form $r_2 \equiv (b^q a^\varepsilon z_2 a^\eta)^{n_2}$ where $n_2 \geq 1$, $\varepsilon, \eta = \pm 1$ and z_2 is b^q -free.
- ④ $|A| \geq 3$ and there exist $a, b \in A$ such that
 - (i) $\Theta_R(a^p) = \Theta_R(b^q) = 2$;
 - (ii) the unique relator r involving a^p is of the form $r \equiv (a^p b^\varepsilon z b^\eta)^{n_1}$ where $n_1 \geq 1$, $\varepsilon, \eta = \pm 1$ and z is a^p -free.
- ⑤ $A = \{a, b\}$ and $\Theta_R(a^p) = 2$.

Example

Consider the subcase when $A = \{a, b\}$, $\Theta_R(a^p) = 2$, $\Theta_R(ab) = 2$ and the unique relator r involving a^p has the form $r = (a^p b^{k_1} a z' a^t b^{-k_2})^{k'}$, where $k_1, k_2, k' \geq 1$, $t \in \{1, -1\}$ and $az'a^t$ is a^p -free.

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Put $u = b^{-1}a^{-1}b$ and $w = (a^p)^L$.

Observe that if a^{-1} occurs in $z'a^t$ then $a^s z'a^t$ has a subword of the form $a\tilde{z}a^{-1}$ where $\tilde{z} = b^{\pm l}$ and $1 \leq l \leq q$. But this implies that $\Theta_R(ab) = 1$, which is a contradiction. Similarly, if in any relator $r' \neq r$ there is a subword of the form $az''a^{-1}$ (or equivalently $a^{-1}z''a$) then $\Theta_R(ab) = 1$; so we can assume that in every relator of R all the exponents of the powers of a have the same sign. It follows that the possible labels for an edge $e \in (\mathcal{C}, \mathcal{G})$ are subwords, up to inversion, of a^p , $a^{p-1}b^{-1}$, $b^{-1}a^{-1}$, $a^{-1}b$ and ba^{p-1} . Thus $\Theta_R(\Phi(e)) \leq 2$ for all $(\mathcal{C}, \mathcal{G})$ -edges in \mathcal{M} (in particular \mathbf{A}_5 is satisfied). In this case one can show that $\tilde{d}(e, \Delta) \leq 0$ for all $e \in (\mathcal{C}, \mathcal{G})$ and that $\tilde{d}(e, \Delta) \leq -\frac{1}{6}$ when $\Theta_R(\Phi(e)) = 1$ or there exists a vertex of degree ≥ 4 in the \mathcal{G} -region sharing e with Δ .



Example

Lemma

Let u and w be chosen as above. Then for any \mathcal{C} -region Δ we have $\sum_{e \in \partial\Delta} \tilde{d}(e, \Delta) < -6$.

Proof It follows from our choice of L and from the fact that there are no positive $(\mathcal{C}, \mathcal{G})$ -edges in \mathcal{M} that it is enough to show that for each occurrence of u in $\Phi(\Delta)$ there exists a negative edge. Let e_m be an edge whose label involves the occurrence of b in u ; then $\Phi(e_m) \in \{b, a^{-1}b, ba^l \mid 1 \leq l \leq p-1\}$. We can assume that $\Theta_R(\Phi(e_j)) = 2$, $d(\lambda(e_j)) = d(\mu(e_j)) = 3$ for all $j \in \{m-1, m, m+1\}$, otherwise the result follows immediately. If $\Phi(e_m) = a^{-1}b$ then $\Phi(e_{m+1}) = a^p$ and $\Phi(e_{m-1}) = a^l b^{-1}$ for some $1 \leq l \leq p-1$. Since neither a^p nor $a^{-1}b$ can be R -pieces it follows from the form of the relator r that $\Phi(e_{m+1,2})$ begins with b and that $\Phi(e_{m,2})$ begins with a .



Example

This implies that $a^{-1}b^2$ and $a^{-1}ba^{-1}$ are subwords of $\Phi(\Delta_m)^{-1}$ and $\Phi(\Delta_{m-1})$, respectively, and so $a^{-1}b$ is an R -piece, a contradiction. If $\Phi(e_m) = ba^l$, with $1 \leq l \leq p-1$, then $\Phi(e_{m+1}) = a^p$ and $\Phi(e_{m+1,2})$ begins with b . Thus $ba^l b$ is a subword of $\Phi(\Delta_m)^{-1}$ implying that ab is an R -piece, a contradiction.



Thanks



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The End

