Universal Algebraic Geometry: Equational Domains and Co-Domains

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based on joint results with
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Outline

Domains in Commutative Algebra

A commutative associative ring A, $A \neq 0$, is called a domain if it has no zero-divisors.

Zero-divisors are non-zero elements $a, b \in A$ with $a \cdot b = 0$.

Domains in Classical Algebraic Geometry

Let k be a field.

Theorem 1

For any positive integer n and any algebraic sets $Y, Z \subseteq k^n$ over k the set $Y \cup Z$ is algebraic over k.

Theorem 2

Let Y be an algebraic set over k. Then

$$Y \text{ is irreducible} \iff k[Y] \text{ is a domain.}$$

UNIVERSAL ALGEBRAIC GEOMETRY =

= ALGEBRAIC GEOMETRY
OVER ALGEBRAIC STRUCTURES

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Unification theorems in algebraic geometry
Algebra and Discrete Mathematics, **1**, 80–112, 2008.

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Algebraic geometry over algebraic structures II: Foundations

E. Daniyarova, A. Myasnikov, V. Remeslennikov
Algebraic geometry over algebraic structures III: Equationally Noetherian property and compactness
Southeast Asian Bulletin of Mathematics, **35**, 2011.

E. Daniyarova, A. Myasnikov, V. Remeslennikov
Algebraic geometry over algebraic structures IV: Equational domains and co-domains
Algebra and Logic, **49 (6)**, 715–756, 2010.

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Algebra and Logic.

Basic Notions of Universal Algebraic Geometry

- $\mathbb{L}$ — a first-order language
- $\mathcal{A} = \langle A \mid \mathbb{L} \rangle$ — an algebraic structure in the language $\mathbb{L}$
- $X = \{x_1, \dots, x_n\}$ — a finite set of variables
- $\mathrm{T}_{\mathbb{L}}(X)$ — the set of all terms in $\mathbb{L}$ with variables in X
- $\mathrm{At}_{\mathbb{L}}(X)$ — the set of all atomic formulas in $\mathbb{L}$ with variables in X:

  $$(t = s), \quad R(t_1, \dots, t_{n_R}),$$

  $t, s, t_1, \dots, t_{n_R}$ — terms; R — predicate from $\mathbb{L}$
- atomic formulas $\varphi \in \mathrm{At}_{\mathbb{L}}(X)$ are called equations in the language $\mathbb{L}$
- any subset $S \subseteq \mathrm{At}_{\mathbb{L}}(X)$ is called a system of equations

Algebraic Set, Radical, Coordinate Algebra

$S \subseteq \mathrm{At}_{\mathbb{L}}(X)$ — a system of equations

Algebraic set over $\mathbb{L}$-structure $\mathcal{A}$

$$\mathrm{V}_{\mathcal{A}}(S) = \{ (a_1, \ldots, a_n) \in A^n \mid \mathcal{A} \models \varphi(a_1, \ldots, a_n) \ \forall \varphi \in S \}$$

The radical of a system S = the radical of an algebraic set $Y = \mathrm{V}_{\mathcal{A}}(S)$

$$\mathrm{Rad}_{\mathcal{A}}(S) = \mathrm{Rad}(Y) = \{ \varphi \in \mathrm{At}_{\mathbb{L}}(X) \mid \mathcal{A} \models \varphi(a_1, \ldots, a_n) \ \forall (a_1, \ldots, a_n) \in Y \}$$

The coordinate algebra of an algebraic set Y

is the $\mathbb{L}$-structure

$$\Gamma(Y) = \langle \mathrm{T}_{\mathbb{L}}(X)/\mathrm{Rad}(Y) \mid \mathbb{L} \rangle$$

with the natural interpretation of symbols from $\mathbb{L}$ on the factor-set $\mathrm{T}_{\mathbb{L}}(X)/\mathrm{Rad}(Y)$.

Equationally Noetherian Algebraic Structures

Definition

An algebraic structure $\mathcal{A}$ is called equationally Noetherian if for any positive integer n and any system of equations

$$S(x_1, \ldots, x_n)$$

there exists a finite subsystem

$$S_0 \subseteq S$$

such that

$$\mathrm{V}_{\mathcal{A}}(S_0) = \mathrm{V}_{\mathcal{A}}(S).$$

Unification Theorem A (classification of irreducible coordinate algebras)

Let $\mathcal{A}$ be an equationally Noetherian algebraic structure in a language $\mathtt{L}$. Then for a finitely generated algebraic structure $\mathcal{C}$ of $\mathtt{L}$ the following conditions are equivalent:

1. $\mathrm{Th}_{\forall}(\mathcal{A}) \subseteq \mathrm{Th}_{\forall}(\mathcal{C})$, i.e., $\mathcal{C} \in$ **Ucl**$(\mathcal{A})$;
2. $\mathrm{Th}_{\exists}(\mathcal{A}) \supseteq \mathrm{Th}_{\exists}(\mathcal{C})$;
3. $\mathcal{C}$ embeds into an ultrapower of $\mathcal{A}$;
4. $\mathcal{C}$ is discriminated by $\mathcal{A}$;
5. $\mathcal{C}$ is a limit algebraic structure over $\mathcal{A}$;
6. $\mathcal{C}$ is an algebraic structure defined by a complete atomic type in the theory $\mathrm{Th}_{\forall}(\mathcal{A})$ in $\mathtt{L}$;
7. $\mathcal{C}$ is the coordinate algebra of an irreducible algebraic set over $\mathcal{A}$ defined by a system of equations in the language $\mathtt{L}$.

Unification Theorem C (classification of all coordinate algebras)

Let $\mathcal{A}$ be an equationally Noetherian algebraic structure in a language $\mathtt{L}$. Then for a finitely generated algebraic structure $\mathcal{C}$ of $\mathtt{L}$ the following conditions are equivalent:

1. $\mathcal{C} \in \mathbf{Qvar}(\mathcal{A})$, i.e., $\mathrm{Th}_{\mathrm{qi}}(\mathcal{A}) \subseteq \mathrm{Th}_{\mathrm{qi}}(\mathcal{C})$;
2. $\mathcal{C} \in \mathbf{Pvar}(\mathcal{A})$;
3. $\mathcal{C}$ embeds into a direct power of $\mathcal{A}$;
4. $\mathcal{C}$ is separated by $\mathcal{A}$;
5. $\mathcal{C}$ is a subdirect product of finitely many limit algebraic structures over $\mathcal{A}$;
6. $\mathcal{C}$ is an algebraic structure defined by a complete atomic type in the theory $\mathrm{Th}_{\mathrm{qi}}(\mathcal{A})$ in $\mathtt{L}$;
7. $\mathcal{C}$ is the coordinate algebra of an algebraic set over $\mathcal{A}$ defined by a system of equations in the language $\mathtt{L}$.

The Zariski Topology and Irreducible Algebraic Sets

Definition

The Zariski topology on A^n is the topology with the family of algebraic subsets $Y \subseteq A^n$ as a prebase of closed sets.

FACT: $\mathcal{A}$ is equationally Noetherian $\Longleftrightarrow$ the Zariski topology on A^n is Noetherian for every positive integer n.

Definition

An algebraic set Y is called irreducible if it is not a finite union of proper algebraic subsets.

Equational Domains and Co-Domains

Equational Domains

Theorem 1

For any positive integer n and any algebraic sets $Y, Z \subseteq k^n$ over a field k the set $Y \cup Z$ is algebraic over k.

Theorem 2

Let Y be an algebraic set over a field k. Then

$$Y \text{ is irreducible} \iff k[Y] \text{ is a domain.}$$

Equational Domains

Definition

An algebraic structure $\mathcal{A}$ is called an equational domain if for every positive integer n and every algebraic sets $Y, Z \subseteq A^n$ over $\mathcal{A}$ the set $Y \cup Z$ is algebraic over $\mathcal{A}$.

Theorem

Let $\mathcal{A}$ be an equational domain in a functional language L and Y an algebraic set over $\mathcal{A}$. Then

$$Y \text{ is irreducible} \iff \Gamma(Y) \text{ is an equational domain.}$$

Equational Domains and Co-Domains

Domain

An algebraic structure $\mathcal{A}$ is called an equational domain if for every positive integer n and every algebraic sets $Y_1, \ldots, Y_m \subseteq A^n$ over $\mathcal{A}$ the set $Y_1 \cup \ldots \cup Y_m$ is an algebraic over $\mathcal{A}$.

DOMAINS: every non-empty closed in the Zariski topology set is algebraic.

Co-domain

An algebraic structure $\mathcal{A}$ is called an equational co-domain if for every positive integer n and every algebraic sets $Y_1, \ldots, Y_m \subseteq A^n$ over $\mathcal{A}$ the set $Y_1 \cup \ldots \cup Y_m$ is not an algebraic over $\mathcal{A}$ (if the union $Y_1 \cup \ldots \cup Y_m$ is proper).

CO-DOMAINS: every non-empty algebraic set is irreducible.

Theorem

For an arbitrary algebraic structure $\mathcal{A}$ in a functional language $\mathtt{L}$ the following conditions are equivalent:

- $\mathcal{A}$ is an equational domain;
- every non-empty closed in the Zariski topology set is algebraic;
- there exists a system of equations $S(x_1, x_2, x_3, x_4)$, such that $\mathcal{A}$ satisfies to (infinite) universal formula

$$\varphi = \forall\, x_1, x_2, x_3, x_4 \left(\bigwedge_{(t=s)\in S} t = s \longleftrightarrow [(x_1 = x_2) \vee (x_3 = x_4)] \right).$$

Criteria for Domains

Theorem

For an arbitrary algebraic structure $\mathcal{A}$ in a functional language L the following conditions are equivalent:

- $\mathcal{A}$ is an equational domain;
- every non-empty closed in the Zariski topology set is algebraic;
- there exists a system of equations $S(x_1, x_2, x_3, x_4)$, such that $\mathcal{A}$ satisfies to (infinite) universal formula

$$\varphi = \forall x_1, x_2, x_3, x_4 \left(\bigwedge_{(t=s)\in S} t = s \longleftrightarrow [(x_1 = x_2) \vee (x_3 = x_4)] \right).$$

REMARK: If $\mathcal{A}$ is equationally Noetherian then φ have to be finite! In this case we have

THEOREM:

$\{$axioms for $\mathbf{Ucl}(\mathcal{A})\} = \{$axioms for $\mathbf{Qvar}(\mathcal{A})\} \cup \{\varphi\}$.

Theorem

For an arbitrary algebraic structure $\mathcal{A}$ the following conditions are equivalent:

- $\mathcal{A}$ is an equational co-domain;
- every non-empty algebraic set over $\mathcal{A}$ is irreducible;
- every non-trivial finitely generated algebraic $\mathtt{L}$-structure $\mathcal{C}$ which is separated by $\mathcal{A}$ is also discriminated by $\mathcal{A}$.

Theorem

For an arbitrary algebraic structure $\mathcal{A}$ the following conditions are equivalent:

- $\mathcal{A}$ is an equational co-domain;
- every non-empty algebraic set over $\mathcal{A}$ is irreducible;
- every non-trivial finitely generated algebraic $\mathbb{L}$-structure $\mathcal{C}$ which is separated by $\mathcal{A}$ is also discriminated by $\mathcal{A}$.

If $\mathcal{A}$ is equationally Noetherian then conditions above are equivalent to:

- $\mathbf{Qvar}(\mathcal{A}) = \mathbf{Ucl}(\mathcal{A})_{\mathbf{e}}$;
- $\mathcal{A}\times\mathcal{A} \equiv_{\forall} \mathcal{A}$.

Domains in Rings, in Groups, in Algebras over a Ring

Equational Domains in Commutative Associative Rings

Equational domain:

$$\forall\, x, y \left(\bigwedge_{f \in S} f(x, y) = 0 \;\longleftrightarrow\; [(x = 0) \vee (y = 0)] \right)$$

Domain:

$$\forall\, x, y \quad (\; x \cdot y = 0 \;\longleftrightarrow\; [(x = 0) \vee (y = 0)] \;)$$

Equational Domains in Commutative Associative Rings

Equational domain:

$$\forall\, x, y \left(\bigwedge_{f \in S} f(x, y) = 0 \;\longleftrightarrow\; [(x = 0) \vee (y = 0)] \right)$$

Domain:

$$\forall\, x, y \quad (\; x \cdot y = 0 \;\longleftrightarrow\; [(x = 0) \vee (y = 0)] \;)$$

Statement

Let A be a commutative associative ring with 1. Then the following conditions are equivalent:

1. A is a domain;
2. A is an equational domain in the language $\mathrm{L_r} = \{+, -, \cdot, 0, 1\}$;
3. A is an equational domain in the language $\mathrm{L}_{\mathrm{r}A} = \mathrm{L_r} \cup \{a, a \in A\}$.

Domains in Groups

Definition (G. Baumslag, A Myasnikov, V. Remeslennikov, 1999)

A group G is called a domain if it has no zero-divisors.

An element $1 \neq x \in G$ is called a zero-divisor if there exists $1 \neq y \in G$, such that $[x, y^g] = 1$ for all $g \in G$.

Domains in Groups

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A group G is called a domain if it has no zero-divisors.

An element $1 \neq x \in G$ is called a zero-divisor if there exists $1 \neq y \in G$, such that $[x, y^g] = 1$ for all $g \in G$.

Statement

Let G be a group. Then the following conditions are equivalent:

1. G is a domain;
2. G is an equational domain in the language $\mathrm{L}_{\mathrm{gr}G} = \{\cdot, ^{-1}, 1\} \cup \{g, g \in G\}$.

Let K be a commutative associative ring with 1 and A an algebra over K.

Definition

We name A a domain if it has no zero-divisors.

We name an element $0 \neq a \in A$ a zero-divisor if there exists an element $0 \neq b \in A$, such that

$$\mathrm{id}\langle a \rangle \cdot \mathrm{id}\langle b \rangle = 0 \quad \text{and} \quad \mathrm{id}\langle b \rangle \cdot \mathrm{id}\langle a \rangle = 0.$$

Definition

We name A a domain if it has no zero-divisors.

We name an element $0 \neq a \in A$ a zero-divisor if there exists an element $0 \neq b \in A$, such that

$$\mathrm{id}\langle a\rangle \cdot \mathrm{id}\langle b\rangle = 0 \quad \text{and} \quad \mathrm{id}\langle b\rangle \cdot \mathrm{id}\langle a\rangle = 0.$$

Statement

Let A be an algebra over a commutative associative ring K. Then the following conditions are equivalent:

1. A is a domain;
2. A is an equational domain in the language

$$\mathrm{L}_{K\mathrm{alg}A} = \{+, -, \cdot, 0, \alpha\cdot, \alpha \in K\} \cup \{a, a \in A\}.$$

Zero-Divisors

Elements $0 \neq a, b \in A$ are zero-divisors if

$$\mathrm{id}\langle a\rangle \cdot \mathrm{id}\langle b\rangle = \mathrm{id}\langle b\rangle \cdot \mathrm{id}\langle a\rangle = 0.$$

LIE ALGEBRAS: $[\mathrm{id}\langle a\rangle, b] = 0.$

ASSOCIATIVE ALGEBRAS WITH 1:

$$acb = bca = 0 \quad \text{for all} \quad c \in A.$$

COMMUTATIVE ASSOCIATIVE ALGEBRAS: $a \cdot b = 0.$

FOR ALL ABOVE: domain = prime algebra.

EXAMPLES

Equational Domains in Groups

1. every non-abelian free group
2. every non-abelian torsion-free hyperbolic group
3. every non-abelian simple group
4. free product $A * B$ of any groups A and B (exception $\mathbb{Z}_2 * \mathbb{Z}_2$)

Equational Domains in Algebras over a Field

1. free algebra, free associative algebra, free commutative algebra, free commutative associative algebra, free non-abelian Lie algebra over a field
2. every simple algebra over a field with non-trivial multiplication
3. every prime algebra
4. free product $A * B$ of Lie algebras over a field if A and B are domains

Equational Co-Domains

1. every torsion free abelian group
2. every torsion-free module over a commutative associative domain
3. $\prod_{i \in I} \mathcal{A}^{(i)}$ for any algebraic structure $\mathcal{A}$ and $|I| = \infty$ $(\mathcal{A}^{(i)} \cong \mathcal{A})$

NOT Equational Domains

1. free semigroup
2. every non-trivial commutative monoid with cancelations
3. every non-trivial abelian, nilpotent, solvable group
4. every non-trivial abelian, nilpotent, solvable Lie algebra