# Uniform Words are Primitive

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# Outline

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- Primitivity rank: A new classification of words
- Some words about the proof

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## Definition

A word $w$ is called **uniform** if for every finite group $G$, $\alpha_G(w)$ has uniform distribution.

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- $\exists$ many similar open problems w.r.t. other types of groups

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**Theorem (P-Parzanchevski, 2011)**

*Conjectures 1 and 2 hold for* $\mathbf{F}_k \ \forall k$

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$w$ is primitive

$\Downarrow$ *Obs*

$\alpha_G(w)$ is uniform $\forall$ finite $G$

$\Downarrow$

$\alpha_{S_n}(w)$ is uniform $\forall n$

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$Prob\left[\alpha_{S_n}(w)(1) = 1\right] = \frac{1}{n} \quad \forall n$

*Thm* (the last statement implies "$w$ is primitive")

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**3**: the smallest rank of a subgroup of $\mathbf{F}_k$ in which $w$ is **not** primitive

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- $w$ not primitive in $\mathbf{F}_k \implies \pi(w) \le k$
- Thus, $\pi(w) \in \{0, 1, 2, \ldots, k\} \cup \{\infty\}$

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# Fixed points in $S_n$

The key result:

**Proposition**

$$Prob\left[\alpha_{S_n}(w)(1) = 1\right] = \frac{1}{n} + \frac{a_w}{n^{\pi(w)}} + O\left(\frac{1}{n^{\pi(w)+1}}\right)$$

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$a_w$ - the number of "witness" subgroups:

$$a_w = \left| \left\{ J \leq \mathbf{F}_k \;\middle|\; \begin{array}{c} w \in J \text{ is not primitive in } J \\ \text{and } rk(J) = \pi(w) \end{array} \right\} \right|$$

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| $\pi(w)$ | Description | $Prob\left[\alpha_{S_n}(w)(1)=1\right]$ | $\mathbb{E}\left[\begin{matrix}\#\text{ fixed points}\\ \text{of }\alpha_{S_n}(w)\end{matrix}\right]$ |
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| $k$ | | $\sim\frac{1}{n}+\frac{a_w}{n^k}$ | $\sim 1+\frac{a_w}{n^{k-1}}$ |
| $\infty$ | $w$ is primitive | $\frac{1}{n}$ | 1 |

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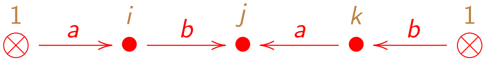


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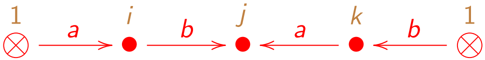


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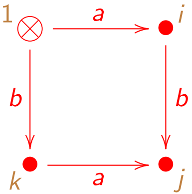
- Case 1: $1, i, j, k$ all distinct:




$$P_1 = \frac{(n-1)(n-2)(n-3)}{n(n-1)\cdot n(n-1)} \in \Theta\left(\frac{1}{n}\right)$$

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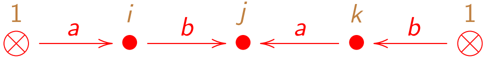


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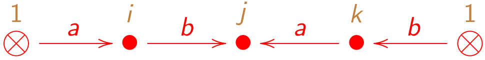


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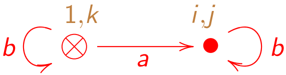
Case 2: $i = j \implies 1 = k$:

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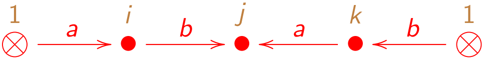


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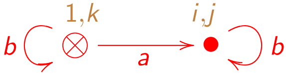


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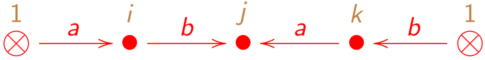


$i, j, k \in [n]$

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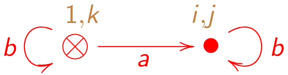
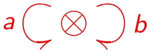



$$P_2 = \frac{(n-1)}{n \cdot n(n-1)} \in \Theta\left(\frac{1}{n^2}\right)$$

Case 3: $1 = i = j = k$:




$$P_3 = \frac{1}{n \cdot n} \in \Theta\left(\frac{1}{n^2}\right)$$

Etc.

# Fixed points in $S_n$

There are 7 cases in total:

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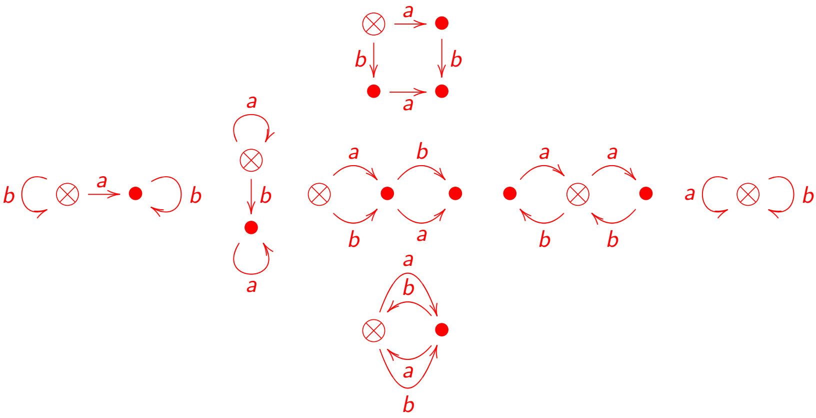

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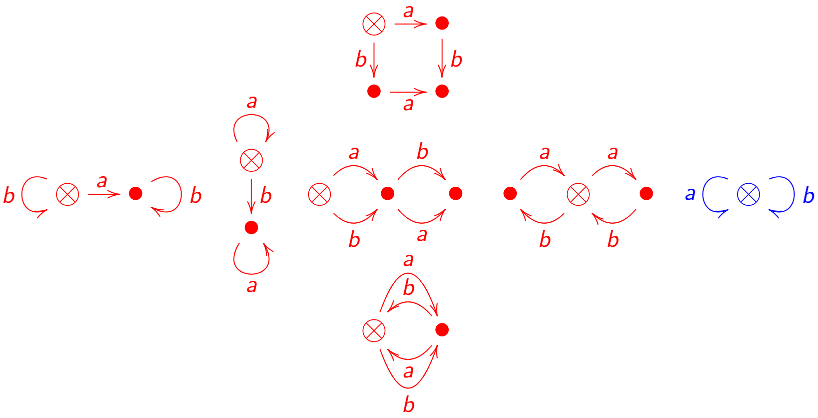

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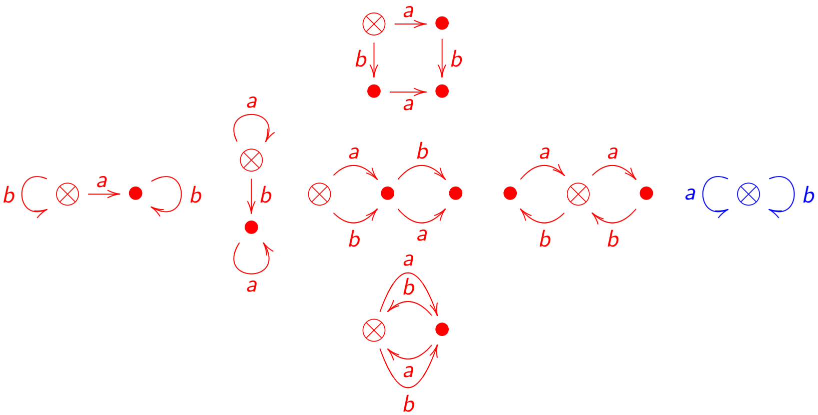


$$Prob\left[\alpha_{S_n}(w)(1) = 1\right] = \frac{1}{n-1}$$

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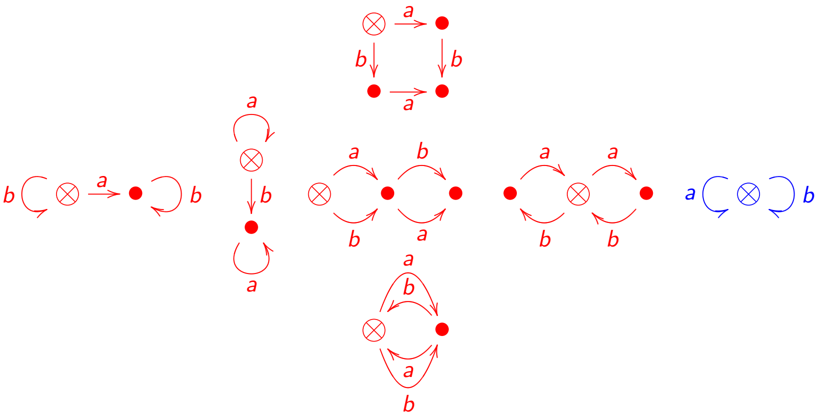

$$Prob\left[\alpha_{S_n}(w)(1) = 1\right] = \frac{1}{n-1} = \frac{1}{n} + \frac{1}{n^2} + O\left(\frac{1}{n^3}\right)$$

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- $w$ is primitive $\overset{?}{\Longleftrightarrow}$ $w$ is uniform w.r.t. compact Lie groups / $U(2)$ / other finite groups ...
- Can we tell apart $w_1, w_2$ from different $Aut(\mathbf{F}_k)$-orbits?
- Understand completely the rational expression $Prob\left[\alpha_{S_n}(w)(1) = 1\right]$

Thank You!