

# Affine actions on non-archimedean trees

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Isometric actions  
on  $\mathbb{R}$ -trees

→

Isometric actions  
on  $\Lambda$ -trees

↓

Affine actions  
on  $\mathbb{R}$ -trees

→

↓

?

Free Isometric actions  
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- Much of the basic theory of isometric actions can be extended to affine actions fairly easily.
- More examples of groups — broader class of groups that can be analysed in this way.
- Brings together under one heading:
  - Lyndon length functions,
  - affine actions,
  - certain 'bad' examples of groups actions.

Let  $\Lambda$  be an **ordered abelian group**; that is, an abelian group written additively, together with a compatible linear order.  
(E.g.  $\mathbb{R}$ ,  $\mathbb{Z} \times \mathbb{Z}$  (lexicographic order),  $\mathbb{Z}^m$ ,  $\mathbb{R}^m$ ,  ${}^*\mathbb{Z}$ .)

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Let  $X$  be a ( $\Lambda$ -)metric space, and  $v \in X$ . Define

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An isometric action on a  $\Lambda$ -tree, and a choice of basepoint  $v$ , gives rise to a **Lyndon length function**  $L$  via  $L : g \mapsto d(v, gv)$ .

A **Lyndon length function**  $L$  satisfies the following where we define  $c(g, h) = \frac{1}{2} (L(g) + L(h) - L(g^{-1}h))$

- 1  $c(g, h) \in \Lambda$ ;
- 2  $L(1) = 0$ ;
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Conversely, if  $L : G \rightarrow \Lambda$  satisfies 1-4, there is a  $\Lambda$ -tree  $X$ , an isometric action of  $G$  on  $X$ , and a basepoint  $v$  such that  $L(g) = d(v, gv)$  for all  $g \in G$ .

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An action is **free** if  $g[x, y] = [x, y]$  implies  $g = 1$ .

Call a group  $\text{ITF}(\Lambda)$  if it admits a free action on some  $\Lambda$ -tree. Call a group  $\text{ITF}$  if it is  $\text{ITF}(\Lambda)$  for some  $\Lambda$ .

## Affine actions on $\mathbb{R}$ -trees

An automorphism  $\phi$  of an  $\mathbb{R}$ -tree is **affine** if there is a constant  $\alpha$  such that  $d(\phi x, \phi y) = \alpha d(x, y)$  for all  $x, y$ .

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### Theorem (I. Liousse, 2001)

*There are groups that admit free affine actions on  $\mathbb{R}$ -trees that don't admit free isometric actions on any  $\mathbb{R}$ -tree. These include*

$$\langle x_1, x_2, \dots, x_n, y_1, y_2, \dots, y_n \mid [x_1, y_1] = [x_2, y_2] = \dots = [x_n, y_n] \rangle,$$

$$\text{and } \langle x_1, x_2, \dots, x_n, y_1, y_2, \dots, y_m \mid w = v^k \rangle$$

where  $w$  is a surface relator (that is, either

$w = [x_1, x_2] \cdots [x_{2r-1}, x_{2r}]$  where  $n = 2r$ , or  $w = x_1^2 \cdots x_n^2$ ) and  $v$  is a non-trivial word in the  $y_i^\pm$ .

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Let  $(X, d)$  be a  $\Lambda$ -metric space. A function  $\phi : X \rightarrow X$  is an **affine automorphism** if there is an order-preserving automorphism  $\alpha = \alpha_\phi$  of  $\Lambda$  such that  $d(\phi x, \phi y) = \alpha(d(x, y))$ .

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We say that a group is **ATF( $\Lambda$ )** (or just ATF) if it admits a free affine action on a  $\Lambda$ -tree.

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$F$  acts isometrically on  $X$  by left multiplication. Define a 'shifting' map  $\tau : F \rightarrow F$  by mapping

$$a_{i_1}^{\epsilon_1} a_{i_2}^{\epsilon_2} \cdots a_{i_m}^{\epsilon_m} \mapsto a_{i_1+1}^{\epsilon_1} a_{i_2+1}^{\epsilon_1} \cdots a_{i_m+1}^{\epsilon_1}.$$

Then  $d(\tau x, \tau y) = 2d(x, y)$  for  $x, y \in F$ , whence  $G = \langle F, \tau \rangle$  has an affine action on  $X$ : the map  $\alpha : G \rightarrow \text{Aut}^+(\mathbb{R})$  is given by  $\gamma \mapsto 2^e$  where  $e$  is the exponent sum of  $\tau$  in  $\gamma$ .

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- Let  $\Lambda$  denote the subset of  $\mathbb{Z} \times \prod_{n \in \mathbb{Z}} \mathbb{Z}$  consisting of those elements  $(n_{-\infty}, \dots, n_{m-1}, n_m, n_{m+1}, \dots)$  with well-ordered support. We give this the lexicographic order: this makes  $\Lambda$  an ordered abelian group.

Define  $\sigma, \tau : X \rightarrow X$  as follows.

$\sigma : (n_m) \mapsto (n'_m)$  where

$$n'_m = \begin{cases} n_{m+1} & \text{for all } m \in \mathbb{Z} \\ n_{-\infty} + 1 & m = -\infty. \end{cases}$$

$\tau : (n_m) \mapsto (n''_m)$  where

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More generally, if  $\Lambda_1, \Lambda_2, \dots, \Lambda_r$  are ordered abelian groups then  $\Lambda_1 \wr \Lambda_2 \wr \dots \wr \Lambda_r$  admits a free affine action on a  $\Lambda$ -tree.

- Let  $X = \Lambda = \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$ ,  
 $\sigma : (n_1, n_2, n_3) \mapsto (n_1 + 1, n_2, n_3 + n_2)$ , and  
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- $[\sigma^k, \tau^l](n_1, n_2, n_3) = (n_1, n_2, n_3 - kl) = [\sigma, \tau]^{kl}(n_1, n_2, n_3)$  and
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In fact, this group is isomorphic to

$\langle \rho, \sigma, \tau \mid [\sigma, \tau] = \rho, [\sigma, \rho] = [\tau, \rho] = 1 \rangle \cong \text{UT}(3, \mathbb{Z})$ , the **discrete Heisenberg group**.

- Fix a positive real number  $a$ . Let  $X = \Lambda = \mathbb{Z} \times \mathbb{R}$ , and  $\sigma : (n_1, n_2) \mapsto (n_1 + 1, an_2)$ ,  $\tau : (n_1, n_2) \mapsto (n_1, n_2 + 1)$ , and let  $G = G_a = \langle \sigma, \tau \rangle$ . Write  $\tau_k = \sigma^{-k} \tau \sigma^k$ . Then

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- If  $a$  is algebraic, then  $G_a$  satisfies an extra relation  $w = 1$  arising from the minimum polynomial of  $a$ . (e.g. if  $a = \frac{1 + \sqrt{5}}{2}$  then the minimum polynomial is  $x^2 - x - 1$ , and the extra relation is  $\tau_0 \tau_1^{-1} \tau_2^{-1} = 1$ ).

- If  $a \in \mathbb{Z}$  then the relations  $C$  are automatic, and

$$G_a \cong \langle \sigma, \tau \mid \sigma\tau\sigma^{-1} = \tau^a \rangle \cong \text{BS}(1, a),$$

a **soluble Baumslag-Solitar group**.

More generally, a free affine action on  $\mathbb{R} \times \mathbb{R}$  is obtained if

$\sigma_a, \tau : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R} \times \mathbb{R}$  are defined via

$\sigma_a : (n_1, n_2) \mapsto (n_1 + \log a, an_2)$  and

$\tau_b : (n_1, n_2) \mapsto (n_1, n_2 + b)$ . Then

$$\langle \sigma_a (a > 0), \tau_1 \rangle \cong \langle \sigma_a (a > 0), \tau_b (b \in \mathbb{R}) \rangle$$

which is isomorphic to the group of matrices of the form

$$\begin{pmatrix} e^a & 0 & b \\ 0 & 1 & a \\ 0 & 0 & 1 \end{pmatrix} \quad a, b \in \mathbb{R}.$$

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Conversely, if  $L : G \rightarrow \Lambda$  satisfies 1-4, there is a  $\Lambda$ -tree  $X$ , an action of  $G$  on  $X$ , and a basepoint  $v$  such that  $L(g) = d(v, gv)$  for all  $g \in G$ .

Let  $\alpha : G \rightarrow \text{Aut}^+(\Lambda)$  be a homomorphism.

An  $\alpha$ -Lyndon length function is a function  $L : G \rightarrow \Lambda$  satisfying the following where we define

$$c_\alpha(g, h) = \frac{1}{2} ( L(g) + L(h) - \alpha_g L(g^{-1}h) ).$$

- ①  $c(g, h) \in \Lambda$ ;
- ②  $L(1) = 0$ ;
- ③  $L(g) = \alpha_g L(g^{-1})$ ;
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Conversely, if  $L : G \rightarrow \Lambda$  satisfies 1-4, there is a  $\Lambda$ -tree  $X$ , an  $\alpha$ -affine action of  $G$  on  $X$ , and a basepoint  $v$  such that  $L(g) = d(v, gv)$  for all  $g \in G$ .

## Line stabilisers

Recall that if  $G$  has a free **isometric** action on a  $\Lambda$ -tree then end stabilisers are line stabilisers: that is, if  $g\epsilon = \epsilon = h\epsilon$  then there is a unique  $\epsilon' \neq \epsilon$  such that  $g\epsilon' = \epsilon' = h\epsilon'$  also.

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Suppose that  $G$  has a free **affine** action on a  $\Lambda$ -tree where  $\text{Aut}^+(\Lambda)$  is soluble. Then

$$\{\text{end stabilisers}\} = \{\text{line stabilisers}\} = \{\text{maximal soluble subgroups of } G\}$$

Moreover,  $G$  is CSS and soluble-transitive. ('CSS' means non-trivial elements are contained in a unique maximal soluble subgroup of  $G$ , and maximal soluble subgroups are malnormal)

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If  $\text{Aut}^+(\Lambda)$  is (torsion-free) nilpotent, we can replace 'soluble' above by 'nilpotent'.

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... one can take the ultraproduct (with respect to an ultrafilter  $\mathcal{D}$  on the indexing set  $I$ ) of each item above. This gives an  $\alpha$ -affine action of  $G$  on  $X$  with based length function  $L$ , where  $G = \prod_{i \in I} G_i / \mathcal{D}$ ,  $\Lambda = \prod_{i \in I} \Lambda_i / \mathcal{D}$ , etc.

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- If groups  $G_i$  have affine actions on  $\Lambda$ -trees, the free product  $G$  has a natural action extending the actions of  $G_i$ . Moreover, if the given actions are

- 1 free;
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- Bass's main results on isometric actions on  $\mathbb{Z} \times \Lambda_0$ -trees can be generalised to the affine case. As a special case...

## Theorem (cf Bass 4.7)

Let  $(\mathcal{G}, Y^*)$  be a graph of groups where each vertex group  $\mathcal{G}(x^*)$  has a free isometric action on a  $\Lambda_0$ -tree, and each edge group is cyclic and maximal abelian in its respective end vertex groups. Let an end  $\epsilon_e$  of  $X(\partial_0 e)$  of full  $\Lambda_0$ -type be given for each edge  $e \in E(Y^*)$ .

Assume that

- 1 edge groups coincide with end stabilisers in the vertex groups:

$$\iota_e \mathcal{G}(e) = (\mathcal{G}(\partial_0 e))_{\epsilon_e} \text{ for } e \in E(Y^*).$$

- 2 if  $\partial_0 e = \partial_1 e = x^*$  and  $e \neq f$  then  $\epsilon_e$  and  $\epsilon_f$  lie in distinct  $\mathcal{G}(x^*)$  orbits.

Then  $\pi_1(\mathcal{G}, Y^*)$  has a free affine action on a  $\mathbb{Z} \times \Lambda_0$ -tree extending the given actions.

**However** recall that an HNN extension of an ITF( $\Lambda_0$ ) group is *not* necessarily ITF unless the translation lengths of the images of the generator of the edge group can be made to agree.

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In particular,  $\ell[x, y] < \ell[x^2, y^2]$  for any free isometric action (unless  $x$  and  $y$  commute). Therefore the HNN extension

$$\langle x, y, t \mid t^{-1}[x, y]t = [x^2, y^2] \rangle$$

does not admit a free isometric action on any  $\Lambda$ -tree. However, this group does admit a free affine action on a  $\mathbb{Z} \times \mathbb{Z}$ -tree.

More generally, if  $m, n, r$  and  $s$  are non-zero integers, the groups

$$\Gamma(m, n; r, s) = \langle x, y, t \mid t^{-1}[x^m, y^n]t = [x^r, y^s] \rangle$$

are  $\text{ATF}(\mathbb{Z} \times \mathbb{Z})$ .

(They are ITF if and only if  $|m| - |r|$  and  $|s| - |n|$  have the same sign.)

## Some open questions

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This would mean that all finitely generated torsion-free nilpotent groups are  $ATF(\mathbb{Z}^n)$  for some  $n$ .  
This in turn would mean that all **residually torsion-free nilpotent groups** are ATF.

- ③ Connection with  $\Lambda$ -words? (Cf recent work of Chiswell-Müller and Kharlampovich, Myasnikov, Nikolaev, Remeslennikov, Serbin)

Thank you!