Automorphisms of relatively hyperbolic groups

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What to remember from this talk

A one-ended relatively hyperbolic group G has a canonical splitting.

This gives a lot of information about $Out(G) = Aut(G)/Inn(G)$.

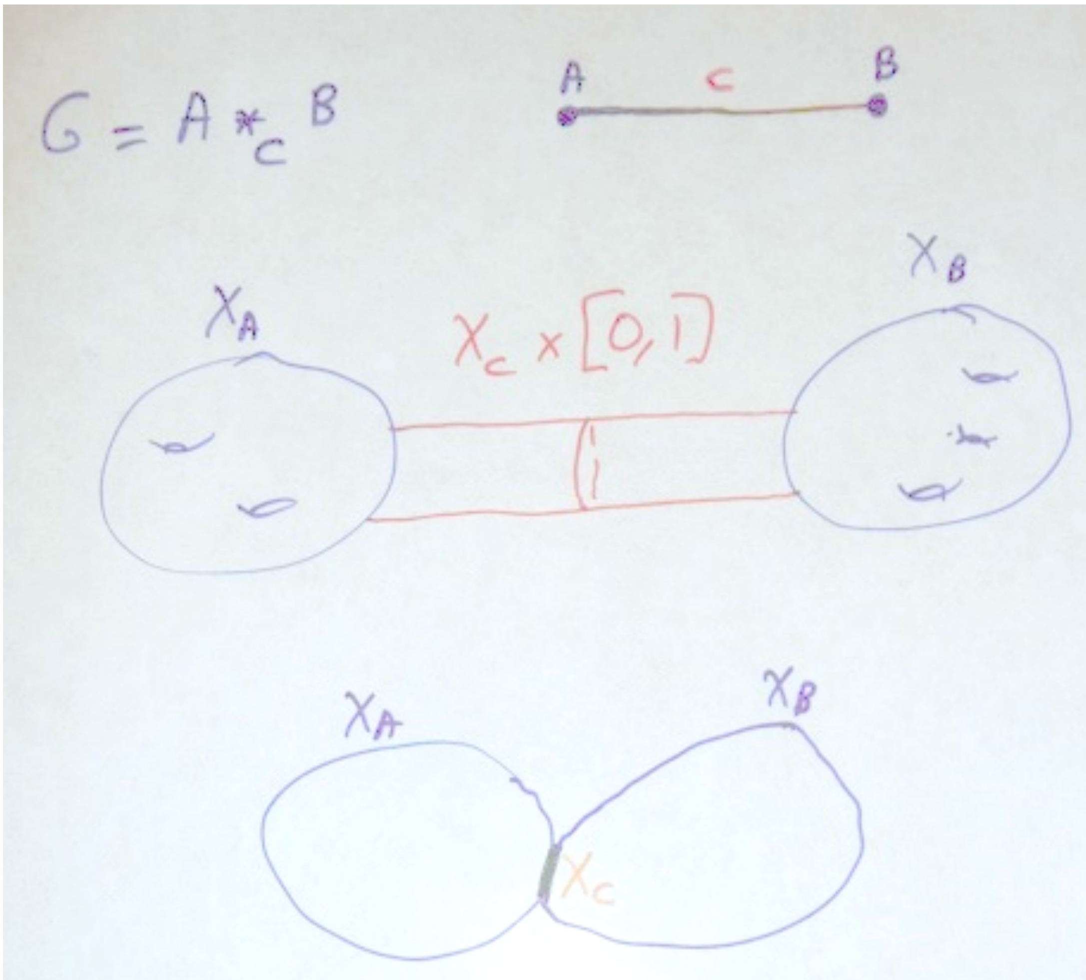
Splittings

A splitting is a decomposition of G as fundamental group of a graph of groups Γ. Equivalently, an action of G on a simplicial tree.

Simplest case: a free product with amalgamation $G = A *_C B$ (splitting over C).

Topologically: an extension of the Seifert - van Kampen theorem describing π_1 of a union from π_1 of the pieces (vertex groups).

$Out(\Gamma) \subset Out(G)$: automorphisms preserving the splitting.



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Elements of $Out(\Gamma)$: vertex automorphisms

If $\varphi \in Aut(A)$ is the identity on C and is not inner, extend it by the identity to an automorphism of $G = A *_C B$: vertex automorphism.

Topologically: extend a homeomorphism of X_A equal to the identity on X_C.

Elements of $Out(\Gamma)$: twists

If $a \in A$ commutes with C, define $\alpha \in Aut(G)$ by:

$\alpha(g) = aga^{-1}$ if $g \in A$

$\alpha(g) = g$ if $g \in B$.

twist around the edge

Example: if a generates $C \simeq \mathbb{Z}$, get Dehn twist.

Fact

If $Out(C)$ is finite, vertex automorphisms and twists virtually generate $Out(\Gamma)$.
True in a graph of groups if all edge groups have finite Out.

So what?

We "understand" $Out(\Gamma)$. But:

- how big is $Out(\Gamma)$? is it the whole of $Out(G)$?
- we need to understand automorphisms of vertex groups.

These problems have fairly satisfactory answers for relatively hyperbolic groups:

- there is an $Out(G)$-invariant splitting;
- its vertex groups are nice or may be ignored.

Infinitely-ended groups

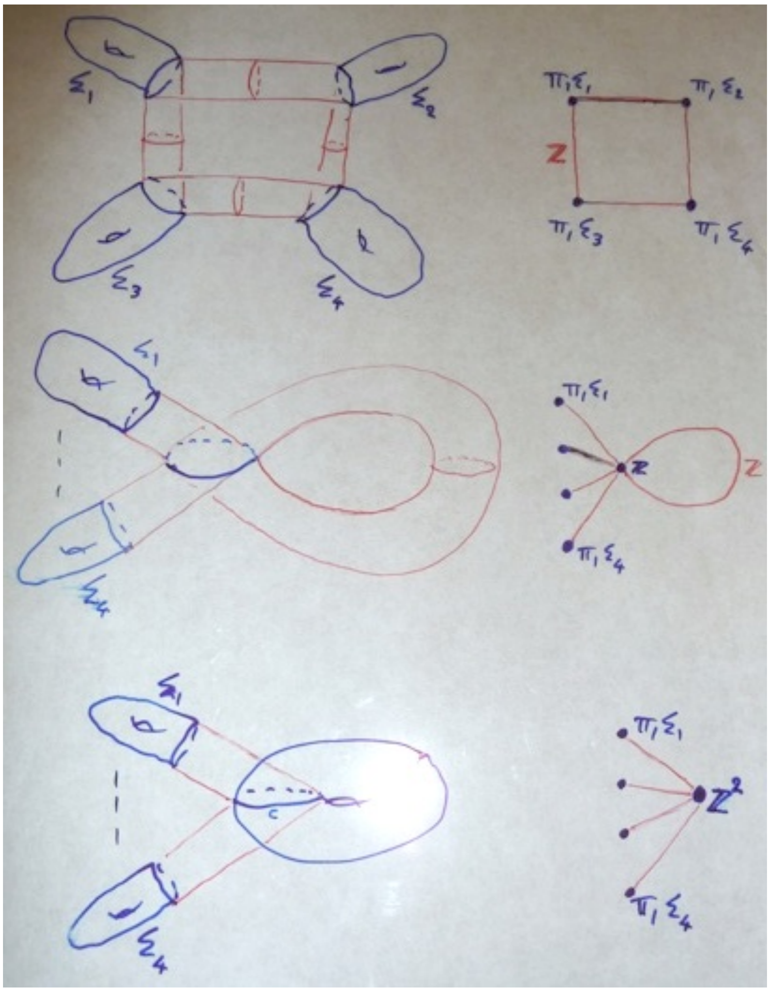
Two kinds of finitely generated groups: infinitely many ends, one end (groups with 0 or 2 ends have finite Out, so forget about them).

Infinitely-ended groups: free groups, free products, all groups splitting over a finite group C.

They don't have canonical splittings. Study $Out(G)$ by letting it act on spaces of splittings (contractible complexes).

Basic example: Culler-Vogtmann's outer space for $Out(F_n)$.

We therefore consider one-ended groups (don't split over a finite group).



Relatively hyperbolic groups

$G = \pi_1(X)$ is one-ended, torsion-free. It is not (Gromov)-hyperbolic, because it contains $\mathbb{Z}^2$, but it is hyperbolic relative to this subgroup $P = \mathbb{Z}^2$ (parabolic subgroup).

Relatively hyperbolic groups generalize π_1's of complete hyperbolic manifolds with finite volume. Such a manifold consists of a compact part and cusps. Its π_1 acts properly on $\mathbb{H}^n$, the action is cocompact after removing horoballs coming from the cusps.

To define a general relatively hyperbolic group, replace $\mathbb{H}^n$ by a proper δ-hyperbolic space. Maximal parabolic subgroups are stabilizers of points in the boundary.

$Out(G)$ from an invariant splitting (example)

The first splitting is not $Out(G)$-invariant: cannot swap $\pi_1(\Sigma_1)$ and $\pi_1(\Sigma_2)$.

The second splitting is better, but not perfect: the automorphism conjugating $\pi_1(\Sigma_1)$ by the class of γ (going around the torus) does not preserve the splitting.

The third splitting is $Out(G)$-invariant, so we can use it to describe $Out(G)$.

$Out(G)$ from an invariant splitting (example)

Some finite index $Out^0(G) \subset Out(G)$ fits in a short exact sequence

$$1 \to \mathbb{Z}^6 \to Out^0(G) \to \mathbb{Z} \times \prod_{i=1}^{4} MCG(\Sigma_i) \to 1.$$

$\mathbb{Z}^6$ is generated by twists; the product comes from vertex automorphisms; $\mathbb{Z}$ comes from vertex automorphisms at the parabolic subgroup $\mathbb{Z}^2 = \langle c, \gamma \rangle$ fixing c.

$Out(G)$ from an invariant splitting

Theorem (Guirardel-L.)

G toral relatively hyperbolic (torsion-free, hyperbolic relative to $\mathbb{Z}^k$ subgroups), one-ended. There is an exact sequence

$$1 \to \mathbb{Z}^p \to Out^0(G) \to \prod_{i=1}^{q} GL(m_i, n_i, \mathbb{Z}) \times \prod_{j=1}^{r} MCG(\Sigma_j) \to 1$$

with $GL(m_i, n_i, \mathbb{Z})$= automorphisms of $\mathbb{Z}^{m_i+n_i}$ equal to the identity on $\mathbb{Z}^{m_i}$ (block-triangular matrices).

Vertex groups of the invariant splitting are maximal parabolic subgroups, surface groups, or rigid. Rigid groups have finite (relative) Out (follows from standard arguments: Bestvina, Paulin, Rips, Belegradek-Szczepański) so they may be absorbed in Out^0.

What next?

Construction of the canonical splitting [Guirardel-L.]:

JSJ theory provides the starting point. The invariant splitting is obtained by the "tree of cylinders" construction. The parabolic subgroups become elliptic (contained in a vertex group).

Applications:

- Finiteness properties (finite presentability, VFL) [Guirardel-L.]:
  • for $Out(G)$, with G toral relatively hyperbolic (possibly infinitely-ended).
  • if $H \subset F_n$ is finitely generated, malnormal, then $\widetilde{Out}(H) \subset Out(H)$, consisting of automorphisms extending to F_n, is VFL. By malnormality, F_n is hyperbolic relative to H (Bowditch). Uses JSJ over non-small groups. What if H not malnormal?

- Residual finiteness of $Out(G)$ for G one-ended, hyperbolic relative to small, residually finite, subgroups. [L.-Minasyan]
- Characterization of relatively hyperbolic groups (possibly infinitely-ended) with $Out(G)$ infinite. [Guirardel-L.]

Residual finiteness

A group H is residually finite if it has a lot of finite index subgroups (equivalently: a lot of finite quotients): given $\Phi \neq 1$, there is $\pi : H \to F$ with F finite and $\pi(\Phi) \neq 1$.

Implies solution to the word problem, Hopfianity,...

Open question: is every hyperbolic group residually finite?

Let's see why $Out(G)$ is residually finite in the example.

Proving residual finiteness on the example (1)

Recall the extension

$$1 \to \mathbb{Z}^6 \to Out^0(G) \xrightarrow{\rho} \mathbb{Z} \times \prod_{i=1}^{4} MCG(\Sigma_i) \to 1$$

coming from the invariant splitting.

Enough to show $Out^0(G)$ residually finite. Given $\Phi \in Out^0 G)$, OK if $\rho(\Phi) \neq 1$, so assume Φ is a product of (powers of) twists; we're not done because the extension is not a product.

Idea: make edge groups of the splitting finite.

Proving residual finiteness on the example (2)

Fix N large, and kill c^N. Get $G_N = G/\langle\langle c^N \rangle\rangle$ with a similar graph of groups structure: Σ_i becomes a closed orbifold with a conical point of order N, edge groups are replaced by $\mathbb{Z}/N\mathbb{Z}$, and $\mathbb{Z}^2$ is replaced by $\mathbb{Z}/N\mathbb{Z} \times \mathbb{Z}$.

$G \twoheadrightarrow G_N = G/\langle\langle c^N \rangle\rangle$ induces $Out(G) \to Out(G_N)$. If N is large, Φ maps non-trivially.

What have we gained? G_N has infinitely many ends!

Theorem (Minasyan-Osin)

If H has infinitely many ends and is residually finite, then $Out(H)$ is residually finite.

This completes the proof for the example since $Out(G_N)$ is residually finite.

Theorem (L.-Minasyan)

If G is one-ended, hyperbolic relative to small, residually finite, subgroups, then $Out(G)$ is residually finite.

Corollary

The following are equivalent:

1. *Every hyperbolic group G is residually finite.*
2. *Every hyperbolic group G has a proper subgroup of finite index.*
3. *Every hyperbolic group G has $Out(G)$ residually finite.*

$1 \iff 2$ by Kapovich-Wise (2000).

$1 \implies 3$ by theorem if G one-ended, by Minasyan-Osin if infinitely-ended.

$3 \implies 1$ by general fact: $G \hookrightarrow Out(G * F_2)$.

Constructing the invariant splitting as a tree of cylinders

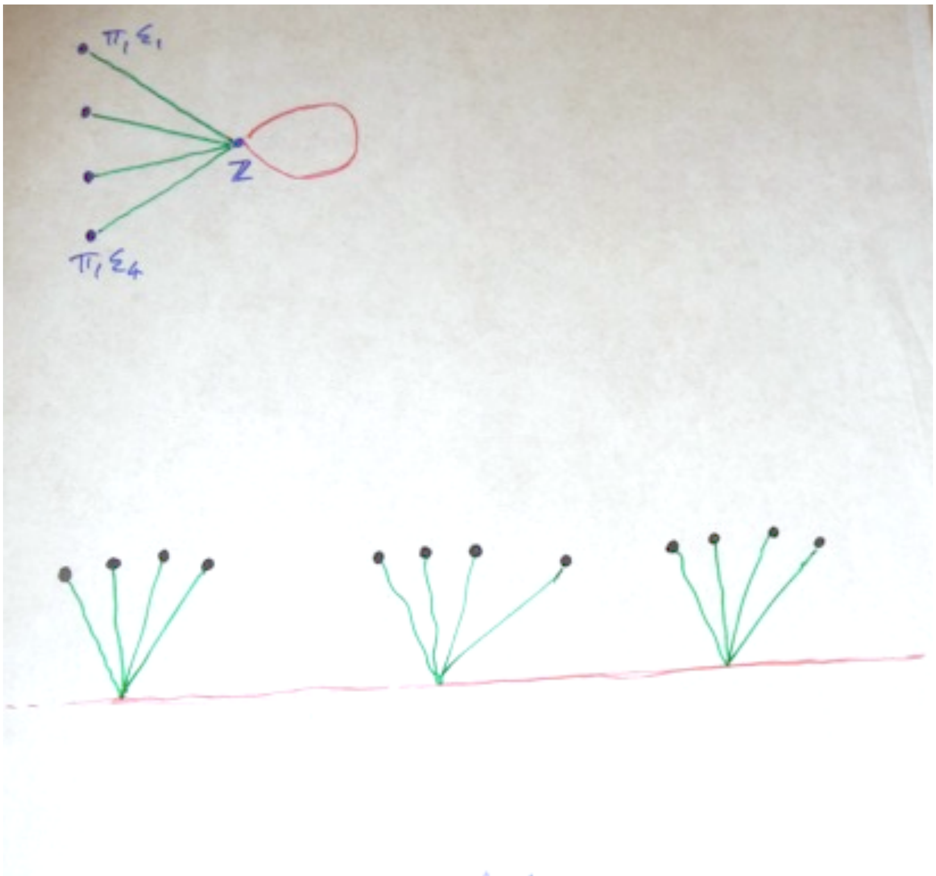
For simplicity: G toral relatively hyperbolic, one-ended.

Use as starting point a JSJ splitting over abelian (loxodromic or parabolic) subgroups (one of the first two splittings). The third splitting is its tree of cylinders.

Say that two edges of the Bass-Serre tree are in the same cylinder if their stabilizers generate an abelian subgroup. (In the example, edge groups are cyclic, they are in the same cylinder iff they are equal)

Fact: cylinders are subtrees.

Define the tree of cylinders T_c by replacing every cylinder by the cone on its boundary (vertices belonging to at least another cylinder). In example: boundary is black, collapse orange line to a point. (In first splitting, cylinders are lines)



A cylinder

Constructing the canonical splitting (2)

Fact: if two trees have the same elliptic subgroups, they have the same tree of cylinders. Invariance of T_c under $Out(G)$ follows since the set of JSJ splittings is invariant under automorphisms, and all JSJ splittings have the same elliptic subgroups (they belong to the same deformation space).

Price to pay for invariance: T_c has more elliptic subgroups (in more general situations, it may be a point). For relatively hyperbolic groups, the new elliptic subgroups are parabolic (T_c is an abelian JSJ splitting relative to the parabolic subgroups) so the vertex groups of T_c are under control (they are abelian, surface, rigid).