NON-TRIVIAL PSEUDOCHARACTERS ON GROUPS

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Basic definitions

- A quasicharacter on a group *G* is a function *f* from the group *G* to the space of real numbers *R* such that $|f(ab) - f(a) - f(b)| < \varepsilon$ for some positive number ε and for any $a, b \in G$.

- A pseudocharacter is a quasicharacter *f* such that $f(a^n) = nf(a)$ for any $a \in G$.

Basic definitions

- An additive character or just character on a group G is a function such that

  $\phi(ab) = \phi(a) + \phi(b)$ for any elements $a, b \in G$.

- A pseudocharacter such that

  $$\phi(ab) \neq \phi(a) + \phi(b)$$

  for some $a, b \in G$ is called non-trivial.

Applications of pseudocharacters.

The existence of non-trivial pseudocharacters on a group is related to with many important characteristics and properties of groups, such as their groups of cohomologies, width of verbal subgroups, stability equations on groups.

Groups of cohomologies

- All pseudocharacters of an arbitrary group G form a real vector space
- *PX(G)* - the space of pseudocharacters
- *X(G)* - the space of additive characters
- **Theorem (Grigorchuk).** *An isomorphism of vector space*

$$H_{b,2}^{(2)}(G) \cong PX(G)/X(G)$$

holds.

Width of verbal subgroups

If on group G there exists a nontrivial pseudocharacter then **any verbal commutator subgroup V(G), has infinite width** relative to a finite proper set of words V from a derived subgroup G‘.

Derived subgroup G' of a group, on which there exists a nontrivial pseudocharacter, has infinite width relative to commutators.

Theorem (Faiziev).
Free products.

Let A and B be non-identity groups not simultaneously equal to Z2.

Then their free product G = A * B

has a non-trivial pseudocharacter,

null on the set AUB.

Amenable groups

Non-trivial pseudocharacters do not exist on soluble groups.

Non-trivial pseudocharacters do not exist on amenable groups.

All pseudocharacters on amenable groups are additive.

Grigorchuk's results

Theorem 1 (Grigorchuk).

A non-trivial pseudocharacter exists on a free products $(A * B, V)$ of groups A and B with amalgamated subgroup V if the following conditions are satisfied: the number of double cosets $|A :: V| \geq 3$, and V is a proper subgroup of the group B.

Grigorchuk’s results

Theorem 2 (Grigorchuk).

A non-trivial pseudocharacter exists on a HNN-extension $G = < G_o, t \mid tAt^{-1} = B >$ *provided that A and B are proper subgroups of the group Go.*

Corollary (Grigorchuk). *Let G be a group with one defining relation and at least three generators. Then non-trivial pseudocharacters exist on G.*

PSEUDOCHARACTERS ON ANOMALOUS PRODUCTS

Theorem 1.

Let $G = A_W B$ be the anomalous product of groups A and B, where $A = \langle x \rangle_\infty$ is an *infinite cyclic group; B is a group that is not normal closure of its elements, and for which the theorem about freedom holds, and $w = x^{k_1} b1 \ldots x^{k_l} b_l$ is a cyclically reduced element of a free product $F = A * B$. Let also the sum of all powers of the element x, with which it enters into the word w, be equal to 0: $\sum_{i=1}^{l} k_i = 0$. Then the anomalous product $G = A_W B$ has a non-trivial pseudocharacter.*

Locally indicable group

- A group is called locally indicable, if any its finitely generated subgroup has homomorphism on the infinite cyclic group.

- *If A is a finitely generated and locally indicable group, then it decomposes into a semidirect product* $A = A\lambda < x >$.

PSEUDOCHARACTERS ON ANOMALOUS PRODUCTS

Proposition 2 .

Let A and B be locally indicable groups. Suppose that A is not finitely generated, B is not cyclic, and w $\in$ A*B is not conjugate to elements of AUB.

Then the anomalous product G = AwB has a non-trivial pseudocharacter.

PSEUDOCHARACTERS ON ANOMALOUS PRODUCTS

Theorem 3.

Suppose that B is a locally indicable non-cyclic group, $A = \langle x \rangle_\infty$ *and* $w = x^{p_1} b_1 \ldots x^{p_l} b_l$ *is a cyclically reduced element of A*B. Let* $\sum p_i = 0$.

Then the anomalous product $G = A_w B$ *has a non-trivial pseudocharacter.*

Questions raised by Grigorchuk

Question 1 (Grigorchuk). *Let G be a non-amenable group with one defining relation. Is it true that there are non-trivial pseudocharacters on G?*

Question 2 (Grigorchuk). *Let F be a free group of rank > 2, and $\alpha : F \to Fq$ is an isomorphism on its subgroup Fq. Is it true that there exists a nonzero α-invariant (i.e. $f(\alpha(x)) = f(x)$) pseudocharacter on F?*

Groups with one defining relation and non-trivial center.

Theorem 4

Let G be a group with one defining relation and non-trivial center. Then on G there exist nontrivial pseudocharacters, with the exception of the following cases: Group G is a cyclic, free abelian:

$G = \langle t, a \mid ta = at \rangle$; *G is metabelian,*

i.e it has a representation in the form

$G = \langle t, a \mid tat^{-1} = a^{p} \rangle$ *or* G =< t,alt2 = a2 > .

Groups with one defining relation and non-trivial center.

Theorem 5

A group with one defining relation and non-trivial center has non-trivial pseudocharacters if this group is not amenable.

Groups with one defining relation and two generators.

$$G = \langle t, a | r(t,a) = 1 \rangle$$

is reduced to HNN-extension

$$\langle t, a_k, \ldots, a_n | s(a_k, \ldots, a_n) = 1, t a_i t^{-1} = a_{i+1}, i = k, k+1, \ldots, n-1 \rangle$$

$$H = \langle a_k, \ldots, a_n | s(a_k, \ldots, a_n) = 1 \rangle$$

Nontrivial pseudocharacters on free groups

$$Fn = \langle a_0, a_1, \ldots, a_{n-1} \rangle, \quad n > 1$$

$$a_0 \to a_1, \ldots, a_{n-2} \to a_{n-1}, a_{n-1} \to U_0 (a_0, \ldots, a_{n-1})$$

Just such endomorphisms should be considered when addressing the questions of Grigorchuk in complicated cases.

Nontrivial pseudocharacters on free groups

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Nontrivial pseudocharacters on free groups

$$U_0 \equiv U_{01} U_{00} U_{02}$$

Nontrivial pseudocharacters on free groups

Theorem 6.

Let $Fn = \langle a_0, \ldots, a_{n-1} \rangle$ *be the free group of rank n, and on Fn an endomorphism is defined, for which*

$$a_0 \to a_1, \ldots, a_{n-2} \to a_{n-1},\ a_{n-1} \to U_0(a_0, \ldots, a_{n-1}),$$

$U_0 = U_{00}$, *reduced form of* U_0 *begins and ends with the positive powers of* a_0, *and* $f(U_0) = 1$, *then on the free group Fn there exists*

a nontrivial pseudocharacter invariant with respect to the considered endomorphism of this group.