Conjugacy growth of finitely generated groups

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Growth

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Definition

The *conjugacy growth function*, denoted ξ_G, will be the function that counts the number of distinct conjugacy classes inside the ball of radius n centered at the identity.

Motivation

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Proposition

Let G be a finitely generated recursively presented group. Then G has solvable conjugacy problem if and only if ξ_G *is recursive.*

What's known

1. Conjugacy growth of fundamental groups of some classes of manifolds (Babenko, Knieper, Link, Margulis, Sinai).
2. Conjugacy growth of hyperbolic groups (Coornaert-Knieper).
3. Solvable and linear groups have exponential or polynomially bounded conjugacy growth. Furthermore, a solvable or linear group with polynomially bounded conjugacy growth is virtually nilpotent (Breuillard-Cornulier, Breuillard-Cornulier-Lubotzky-Meiri, H).
4. Examples (such as diagram groups, Thompson's group F, and many HNN-extensions) and conjectures suggesting most "natural" groups of exponential growth have exponential conjugacy growth (Guba-Sapir).

Example

Let H be the Heisenberg group

$$H = \mathrm{UT}_3(\mathbb{Z}) \cong \langle x, y, z \mid [x, y] = z, zx = xz, zy = yz \rangle.$$

Then each conjugacy class of length at most n has a representative of the form $x^i y^j z^k$ where $i, j \leq n$ and $k \leq gcd(i, j)$. Furthermore, these representatives are all non-conjugate, and since each has length at most $3n$, we get that $\xi_H(n) \sim \sum_{i,\, j=1}^{n} gcd(i, j) \sim n^2 \log(n)$

Results

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Theorem (H, Osin)

Let G be a group generated by a finite set X, ξ_G the conjugacy growth function of G with respect to X. Then the following conditions hold.

(a) *ξ_G is non-decreasing.*

(b) *There exists $a \geq 1$ such that $\xi_G(n) \leq a^n$ for every $n \in \mathbb{N}$.*

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Conversely, suppose that a function $f : \mathbb{N} \to \mathbb{N}$ satisfies the above conditions (a) and (b). Then there exists a finitely generated group G such that $\xi_G \sim f$.

Results

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When speaking about asymptotic invariants of groups it is customary to ask whether these invariants are *geometric*, i.e. invariant under quasi-isometry (up to some suitable equivalence relation). It turns out that the conjugacy growth function is not a geometric invariant.

Theorem (H, Osin)

There exists a finitely generated group G and a finite index subgroup $H \leq G$ such that H has 2 conjugacy classes while G is of exponential conjugacy growth. In particular, conjugacy growth is not a quasi-isometry invariant.

Small cancelation

Our proof of these theorems relies on the theory of small cancelation over relatively hyperbolic groups developed by Osin, which generalized Olshanskii's approach to small cancelation over hyperbolic groups.

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Given a torsion-free relatively hyperbolic group G with two "suitable" elements s_1 and s_2 , for any $g \in G$ we form a quotient of G by adding the relation $g = W$ for some word W in $\{s_1, s_2\}$. The small cancelation theory gives that W can be chosen such that the quotient group is torsion-free and maintains the relatively hyperbolic structure of G. In particular, the quotient map preserves the parabolic subgroups.

Small cancelation

To get an idea of how to apply these small cancelation quotients, let us consider the following theorem which was one of Osin's main applications of this theory.

Theorem (Osin)

There exists a torsion-free 2-generated group in which all non-trivial elements are conjugate.

Sketch of proof

Let R be a countable torsion-free group in which all non-trivial elements are conjugate (this group can be easily constructed using successive HNN-extensions). Let $F = F(x, y)$ be the free group on two generators, and consider the free product $G(0) = R * F$ (which is hyperbolic relative to R). Enumerate all elements of $G(0)$ as $\{1 = g_0, g_1, ...\}$ and all elements of R as $\{1 = r_0, r_1, ...\}$.

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Now we inductively create a sequence of groups and epimorphisms $G(0) \to G(1) \to ...$ as follows. After we have constructed a group $G(i)$, we take an HNN-extension with a stable letter t which conjugates g_{i+1} to some non-trivial element of R.

Sketch of proof

Then we take a quotient of this *HNN*-extension by adding the relations $t = W_1$, $r_{i+1} = W_2$ where W_1, W_2 are words in x and y satisfying sufficiently strong small cancelation conditions. The resulting group is $G(i+1)$, and since the image of t is inside the image of F, there is a natural quotient map from $G(i)$ to $G(i+1)$.

Let G be the direct limit of this sequence of groups. Since $G(0)$ is generated by $\{x, y, r_1, ...\}$ and the image of each r_i is inside the image of F in $G(i)$, the G will be generated by $\{x, y\}$. Since each g_i is conjugate to an element of R in $G(i)$, all non-trivial elements will be conjugate in the limit group. In fact, each quotient map is injective on R, so R naturally embeds into G.

Modifying the proof

We obtain both theorems by using modified versions of the above construction. We will discuss the modifications needed for the first theorem.

Instead of trying to make all elements conjugate, we want to control the number of conjugacy classes inside each ball. So at the ith step of our construction we fix the desired number of conjugacy classes on the sphere of radius i and make all other elements of the sphere conjugate.

Modifying the proof

The main problem, however, is that the conjugacy relations which we want to add may also produce “unwanted” conjugacy relations between elements we want to keep non-conjugate. More precisely, the problem splits into two parts. When dealing with the sphere of radius i at step i, we have to make sure that:

1) “Unwanted” conjugations do not occur inside the ball of radius $(i-1)$.

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1) "Unwanted" conjugations do not occur inside the ball of radius $(i-1)$.

2) We keep enough non-conjugate primitive elements on spheres of radii $> i$ to continue the construction.

Modifying the proof

To overcome the first difficulty we “attach” a new parabolic subgroup with 2 conjugacy classes to a representative of each conjugacy class which we want to keep inside the ball of radius $(i-1)$.

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The second part of the problem is more complicated and requires us to go deep into small cancelation theory to resolve it. Basically, we construct a set of elements on each sphere which we are able to sufficiently control under small cancelation quotients.

Open questions

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Does every amenable group of exponential growth have exponential conjugacy growth? Is every amenable group with polynomial conjugacy growth virtually nilpotent? (both conjectured by Guba-Sapir).

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Question

Which functions can appear as the conjugacy growth function of a residually finite group?

Question

For which classes of groups is conjugacy growth a quasi-isometry invariant?