The index of

the dual lamination to an $\mathbb{R}$-tree

joint work with Thierry Coulbois and Martin Lustig

Manresa 2011

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- $Out(F_N)$ acts on CV_N and its boundary ∂CV_N

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Notion of geometric tree

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How ?

The map $\mathcal{Q}$ [Levitt-Lustig]

$$\begin{array}{rrcll} \mathcal{Q}_P : & F_N & \to & T & \\ & u & \mapsto & u \cdot P & \qquad (P \in T) \end{array}$$

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- Weaken the topology on $\hat{T}$
- $\hat{T}^{obs} = \hat{T}$ with the observers' topology
- $\mathcal{Q}_P : F_N \to \hat{T}^{obs}$ has a unique continuous

$$\mathcal{Q} : \partial F_N \to \hat{T}^{obs}$$

The dual lamination $L(T)$ and the $\mathcal{Q}$-index of T

$$L(T) = \{(X, Y) \in \partial^2 F_N \mid \mathcal{Q}(X) = \mathcal{Q}(Y)\}$$

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$$\mathrm{ind}_{\mathcal{Q}}(T) = \sum_{[P] \in T/F_N} \max(0, \#\mathcal{Q}^{-1}(P) - 2)$$

The limit set and compact heart

The limit set of T is

$$\Omega = \mathcal{Q}^2(L(T)) \subseteq \overline{T}$$

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- system of partial isometries on the compact heart : $\mathcal{S} = (K_A, A)$

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Theorem (Coulbois-H-Lustig)

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Theorem (Coulbois-H-Lustig)

T is the minimal subtree (for the action of F_N) of $T_{\mathcal{S}}$.

$\mathcal{S} = (K_A, A)$ "encodes" T
We a have now a toolbox to study T

Theorem (Coulbois-H)

Let $T \in \partial CV_N$ a tree with dense orbits. Then :

$$\mathrm{ind}_{\mathcal{Q}}(T) \leq 2N - 2.$$

The equality occurs if and only if the limit set Ω contains T.

		$\text{ind}_{\text{geo}}(T) = 2N - 2$	

		$\mathrm{ind}_{\mathrm{geo}}(T) = 2N - 2$	$\mathrm{ind}_{\mathrm{geo}}(T) < 2N - 2$

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An easy computation gives :
$2\ \mathrm{ind}(\Phi) = \mathrm{ind}_{\mathrm{geo}}(T_\Phi) = \mathrm{ind}_{\mathcal{Q}}\ T_{\Phi^{-1}}$

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Φ induced by a pseudo-Anosov

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Φ^{-1} parageometric

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Levitt type	Ω is totally disconnected $\Updownarrow$ Ω_A is totally disconnected $\Updownarrow$ $\text{ind}_{\mathcal{Q}}(T) < 2N - 2$		**pseudo-Levitt** $T_\Phi, T_{\Phi^{-1}}$