Haagerup property

and

free-by-free groups

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Haagerup property / Gromov a-T-menability

Definition

A conditionally negative definite function on a discrete group G is a function $f\colon G \to \mathbb{C}$ such that for any natural integer n, for any $\lambda_1, \cdots, \lambda_n \in \mathbb{C}$ with $\sum_{i=1}^{n} \lambda_i = 0$, for any $g_1, \cdots, g_n$ in G one has

$$\sum_{i,j} \overline{\lambda}_i \lambda_j f(g_i^{-1} g_j) \leq 0.$$

Definition

The group G satisfies the Haagerup property.

$$\Updownarrow$$

There exists a proper conditionnally negative definite function on G.

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There exists a metrically proper affine isometric action on some Hilbert space.

What is the Haagerup property good for ?

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A LOT OF THINGS . . .
AND EVEN MORE THAN THAT.

What is the Haagerup property good for ?

1. Baum Connes and Novikov conjectures (Guolang Yu, V. Lafforgue).
2. Classification of groups :

$$\text{Abelian} \subset \text{Nilpotent} \subset \text{Solvable} \subset \text{Amenable} \subset \cdots$$

$$\cdots \subset \text{a-T-menable} \subset \cdots$$

3. Negation of Kazhdan's Property (T).

Examples

1. Amenable groups (Bekka - Cherix - Valette).
2. Free groups (Haagerup).
3. Coxeter groups (Bojezko - Januszkiewicz - Spatzier).
4. Groups acting properly isometrically on a CAT(0) cubical complex (see Cherix & al).
5. If G fits into an exact sequence

   $$1 \to \text{ a-T-menable group } H \to G \to \text{ amenable group } K \to 1$$

   then G is a-T-menable (Jolissaint).
6. Any wreath-product $\mathbb{F}_n \wr \mathbb{F}_k$ is a-T-menable (de Cornulier - Stalder - Valette).

Non-examples

A group G which fits into a short exact sequence

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1. $\mathbb{Z}^2 \rtimes \mathrm{SL}(2, \mathbb{Z})$ is not Haagerup (de la Harpe - Valette).
2. For any free subgroup $\mathbb{F}_k$ of $\mathrm{SL}_2(\mathbb{Z})$ the semidirect product $\mathbb{Z}^2 \rtimes \mathbb{F}_k$ is not Haagerup (Burger).

Question

What do we know

about semidirect products

$\mathbb{F}_n \rtimes \mathbb{F}_k$?

The Formanek - Procesi group

Definition

Let n be any integer greater or equal to 2. The n^{th}-group of Formanek - Procesi is the semidirect product $\mathbb{F}_{n+1} \rtimes_\sigma \mathbb{F}_n$ where $\mathbb{F}_n = \langle t_1, \cdots, t_n \rangle$ and $\mathbb{F}_{n+1} = \langle x_1, \cdots, x_n, y \rangle$ are the rank n and rank $n+1$ free groups and $\sigma : \mathbb{F}_n \hookrightarrow \mathrm{Aut}(\mathbb{F}_{n+1})$ is the monomorphism defined as follows :
For $i, j \in \{1, \cdots, n\}$, $\sigma(t_i)(x_j) = x_j$ and $\sigma(t_i)(y) = yx_i$.

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Case $n = 2$:

$$\langle x_i, y, t_j \,;\, t_j^{-1} x_i t_j = x_i,\ t_j^{-1} y t_j = y x_j,\ i, j = 1, 2 \rangle$$

The result

Theorem (G.)

The n^{th}-group of Formanek - Procesi acts properly isometrically on some $(2n+2)$-dimensional CAT(0) cube complex and in particular satisfies the Haagerup property.

Space with walls

Definition (Haglund - Paulin)

A space with walls is a pair $(X, \mathcal{W})$ where X is a set and $\mathcal{W}$ is a family of partitions of X into two classes, called walls, such that for any two distinct points x, y in X the number of walls $\omega(x, y)$ is finite.

Definition

A discrete group acts properly on a space with walls $(X, \mathcal{W})$ if it leaves invariant $\mathcal{W}$ and for some (and hence any) $x \in X$ the function $g \mapsto \omega(x, gx)$ is proper on G.

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Theorem (Haglund - Paulin)

A discrete group G which acts properly on a space with walls satisfies the Haagerup property.

From spaces with walls to cube complexes

Definition

Two walls $(u, u^c) \in \mathcal{W}$ and $(v, v^c) \in \mathcal{W}$ cross if all four intersections $u \cap v$, $u \cap v^c$, $u^c \cap v$ and $u^c \cap v^c$ are non-empty. We denote by $I(\mathcal{W})$ the (possibly infinite) supremum of the cardinalities of finite collections of walls which pairwise cross.

Theorem (Chatterji - Niblo, Nica)

Let G be a discrete group which acts properly on a space with walls $(X, \mathcal{W})$. Then G acts properly isometrically on some $I(\mathcal{W})$-dimensional CAT(0) cube complex. In particular it satisfies the Haagerup property.

Structure of the second Formanek-Procesi group

$G := \mathbb{F}_3 \rtimes_\sigma \mathbb{F}_2$ with :

- $\mathbb{F}_2 = \langle t_1, t_2 \rangle$ the vertical subgroup,
- $\mathbb{F}_3 = \langle x_1, x_2, y \rangle$ the horizontal subgroup,
- $\sigma(t_i)(y) = yx_i$, $\sigma(t_i)(x_j) = x_j$.

$G = \langle x_i, y, t_j \, ; \, t_j^{-1} x_i t_j = x_i, \, t_j^{-1} y t_j = y x_j, \, i, j = 1, 2 \rangle$.

Lemma

The group G admits $\{y, t_1, t_2\}$ as a generating set.

Vertical walls

Definition

The vertical j-block $\mathcal{V}_j$ is the set of all the elements in G which admit $t_j t w$, with t a vertical word and w a horizontal word, as a reduced representative. A vertical j-wall is a left-translate $g(\mathcal{V}_j, \mathcal{V}_j^c)$, $g \in G$.

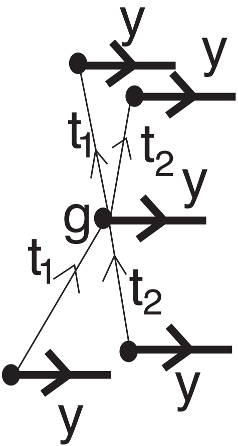
Lemma

The horizontal subgroup is both the left and right G-stabilizer of any vertical wall.

Horizontal walls

Definition

The horizontal block $\mathcal{Y}$ *is the set of all the elements in G which admit tyw, with t a vertical word and w a horizontal word, as a reduced representative. A* horizontal wall *is a left-translate* $g(\mathcal{Y}, \mathcal{Y}^c)$, $g \in G$.



Horizontal walls II

Lemma

The left G-stabilizer of any horizontal wall is a conjugate of the vertical subgroup.

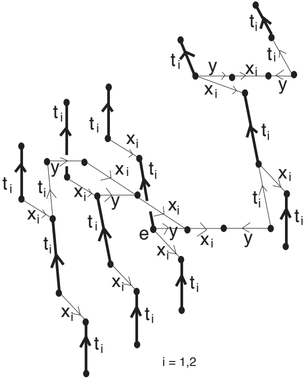
Lemma

Each horizontal wall is invariant under the right-action of the vertical subgroup. More precisely, if $(\mathcal{H}, \mathcal{H}^c)$ is an horizontal wall then for any element t of the vertical subgroup we have $\mathcal{H}t = \mathcal{H}$ and $\mathcal{H}^c t = \mathcal{H}^c$.

Vertizontal blocks

Definition

Let $H_i := \langle x_{i+1}, t_{i+1}, yx_iy^{-1}t_i, x_it_i^{-1}\rangle$ *(*$i = 1, 2 \bmod 2$*), let* $E_i^+ := H_i(e, t_i)$*,* $E_i^- := H_i(t_i, e)$ *and* $E_i := E_i^+ \cup E_i^-$*.*
The vertizontal i*-block* $\mathcal{T}_i$ *is the set of all the elements in* G *which are connected to the identity vertex* e *by an edge-path in* $\Gamma_c \setminus E_i$*.*



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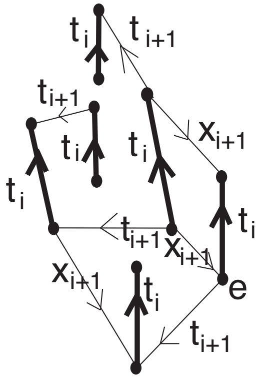
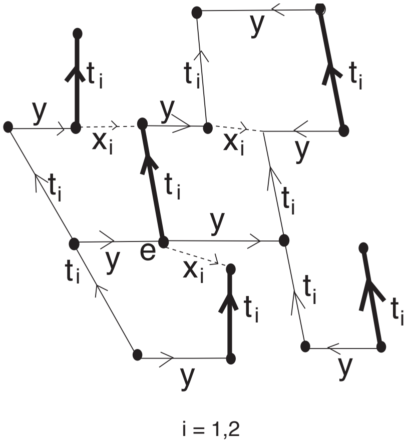


Vertizontal blocks II

Remark

$H_i = \langle y^{-1} t_{i+1}^{-1} y t_{i+1}, t_{i+1}, y t_i y^{-1}, y^{-1} t_i y \rangle$. In particular, for any $k \in \mathbb{Z}$, $y t_i^k y^{-1}$ and $y^{-1} t_i^k y$ are in H_i.



Vertizontal walls

Proposition

There are exactly two connected components in $\Gamma_c \setminus E_i$: the connected component of e and the connected component of t_i.

Definition

A vertizontal i-wall ($i = 1, 2$) is any left-translate $g(\mathcal{T}_i, \mathcal{T}_i^c)$, $g \in G$, of a i-block $\mathcal{T}_i$.

Dimension of the cube complex

Lemma

Let $\mathcal{F}$ be a collection of walls in $(G, \mathcal{W})$ which pairwise cross. Then there is at most one vertical wall and one horizontal wall in $\mathcal{F}$.

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Lemma

For each $i \in \{1, 2\}$:

1. *The vertizontal i-walls $(\mathcal{T}_i, \mathcal{T}_i^c)$ and $y(\mathcal{T}_i, \mathcal{T}_i^c)$ cross.*
2. *There are at most two vertizontal i-wall in $\mathcal{F}$.*

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1. *The vertizontal i-walls $(\mathcal{T}_i, \mathcal{T}_i^c)$ and $y(\mathcal{T}_i, \mathcal{T}_i^c)$ cross.*
2. *There are at most two vertizontal i-wall in $\mathcal{F}$.*

$$\mathcal{F} = \{(\mathcal{Y}, \mathcal{Y}^c), (\mathcal{V}_1, \mathcal{V}_1^c), (\mathcal{T}_i, \mathcal{T}_i^c), y(\mathcal{T}_i, \mathcal{T}_i^c)\,;\, i = 1, 2\}$$

Prospectives

Does any semidirect product $\mathbb{F}_n \rtimes \mathbb{F}_k$ over a free subgroup of polynomially growing outer automorphisms satisfy the Haagerup property ?

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Tentative answer : YES.

Main tool : Bestvina - Feighn - Handel theory for free group automorphisms (“improved relative train-track maps”).