Thompson's group $\mathcal{T}$ as automorphism group of a cellular complex

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(joint work with Maxime Nguyen)

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Obra Social
Fundació "la Caixa"

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Motivation

Tits (1970)

Most subgroups of automorphism groups of trees generated by vertex stabilizers are simple.

Haglund and Paulin (1998)

Most automorphism groups of negatively curved polyhedral complexes are virtually simple.

We are going to be interested in the reciprocal situation:

Given a simple group G (in our case Thompson's group $\mathcal{T}$), find an interesting cellular complex $\mathcal{C}$ such that its automorphism group is 'essentially' the group G.

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What do we already know about $\mathcal{T}$ acting on complexes?

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$\mathcal{F}$, $\mathcal{T}$ and $\mathcal{V}$ act properly and isometrically on CAT(0) cubical complexes.

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There exists a cellular complex $\mathcal{C}$ such that $\mathrm{Aut}_+(\mathcal{C}) \simeq \mathcal{T}$, where $\mathrm{Aut}_+ =$ subgroup of orientation-preserving automorphisms.

Where does the complex $\mathcal{C}$ come from?

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Another viewpoint

- $\Sigma_{g,n}$: compact, connected, orientable surface of genus g with n marked points.
- $\mathrm{MCG}^*(\Sigma_{g,n})$: extended mapping class group of $\Sigma_{g,n}$.
- $\mathcal{C}_c$: curve complex of $\Sigma_{g,n}$.
- $\mathcal{C}_p$: pants complex of $\Sigma_{g,n}$.

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$\mathrm{MCG}^*(\Sigma_{g,n}) \simeq \mathrm{Aut}(\mathcal{C}_c)$, unless $g = 0$ and $n \leq 4$, or $g = 1$ and $n \leq 2$, or $g = 2$ and $n = 0$.

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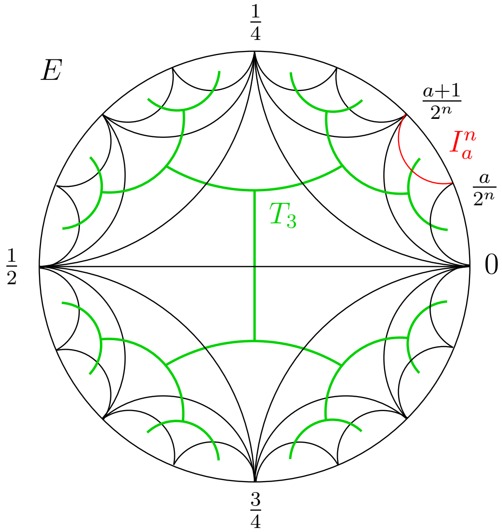
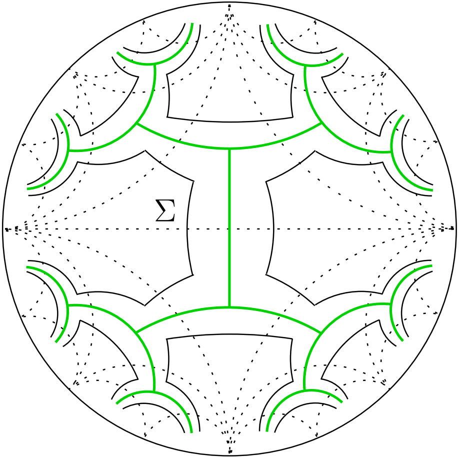
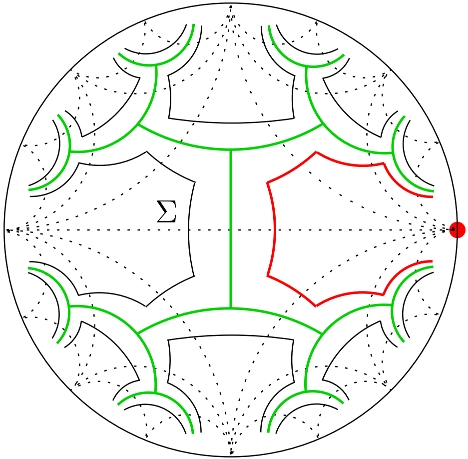
The triangulation E of $\mathbb{D}^2$ and its dual tree



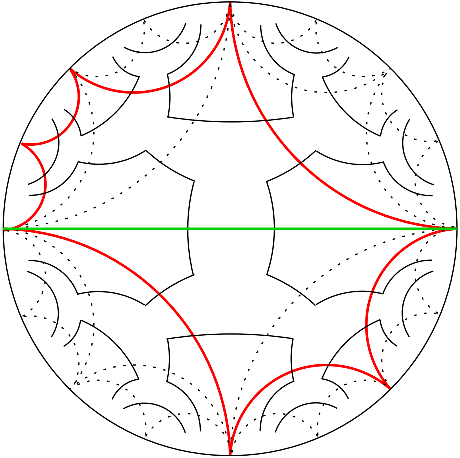
The surface Σ and its hexagonal tessellation



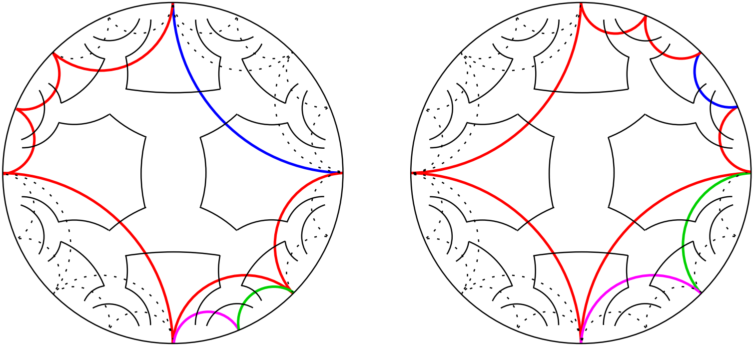
Dyadic rational numbers at the closure of $\partial\Sigma$



Separating arcs: example



Asymptotically rigid homeomorphism: example



The asymptotic mapping class group

$\text{Homeo}_a(\Sigma)$: asymptotically rigid homeomorphisms of Σ.

Definition

AMCG(Σ): quotient of $\text{Homeo}_a(\Sigma)$ by the group of isotopies of Σ.

Proposition

AMCG(Σ) is isomorphic to Thompson's group $\mathcal{T}$.

Definition

Thompson's group $\mathcal{T}$ is the set of orientation preserving piecewise linear homeomorphisms of the circle $S^1 = [0,1]/_{0\sim 1}$ such that:

1. the points of non-differentiability are dyadic rational numbers,
2. the derivatives (where defined) are powers of 2, and
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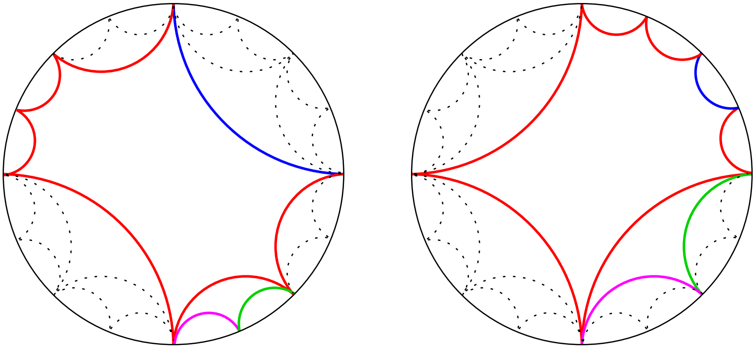
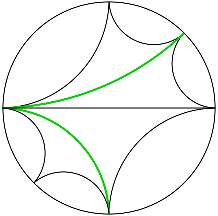
Idea of the proof



Vertices of $\mathcal{C}$



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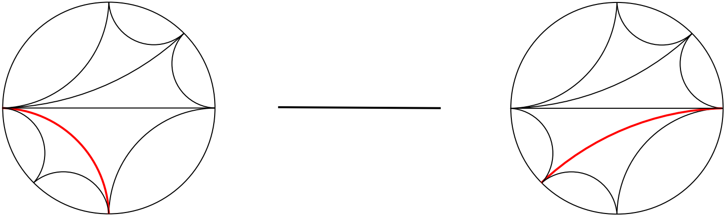
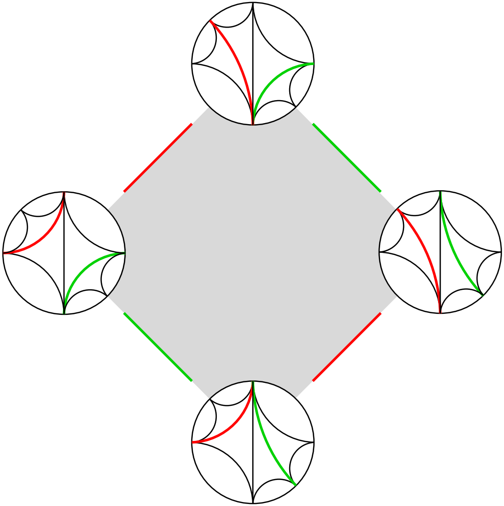
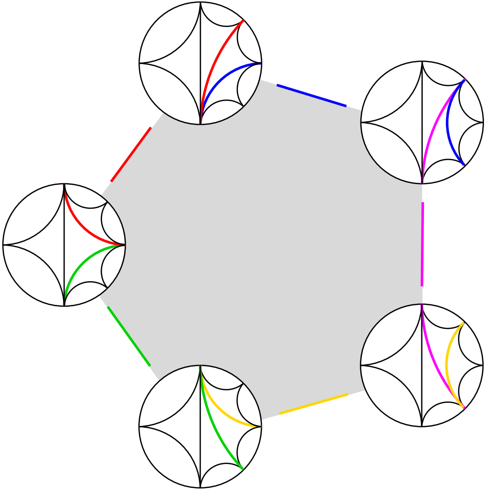


Squared 2-cells of $\mathcal{C}$



Pentagonal 2-cells of $\mathcal{C}$



Some properties of $\mathcal{C}$

1

$\mathcal{C}$ is connected and simply connected.

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$\mathcal{C}$ is locally infinite.
$\{$neighbours of the vertex $v\} \leftrightarrow \{$arcs of the triangulation $v\}$.

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$\mathcal{C}$ is not Gromov hyperbolic.

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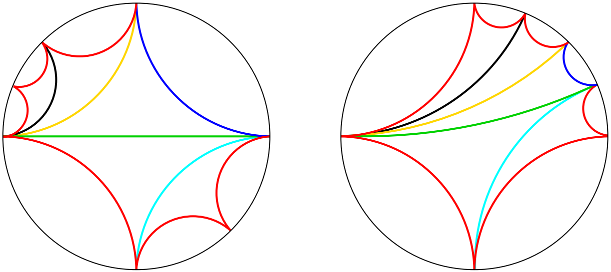
$\mathcal{T}$ acts on $\mathcal{C}$

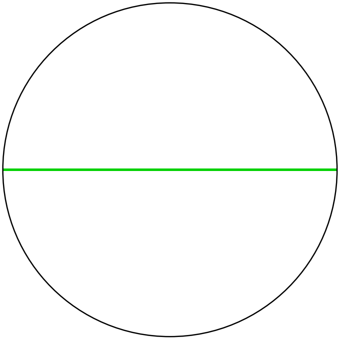
Proposition

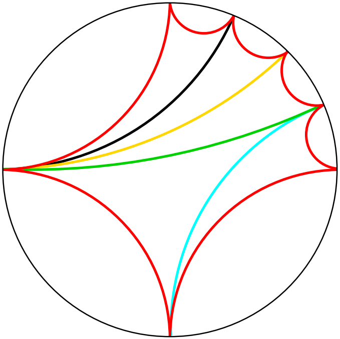
$\mathcal{T}$ acts transitively on $\mathcal{C}$ by automorphisms. Furthermore, the map $\Psi : \mathcal{T} \to \text{Aut}(\mathcal{C})$ is injective.

Action:

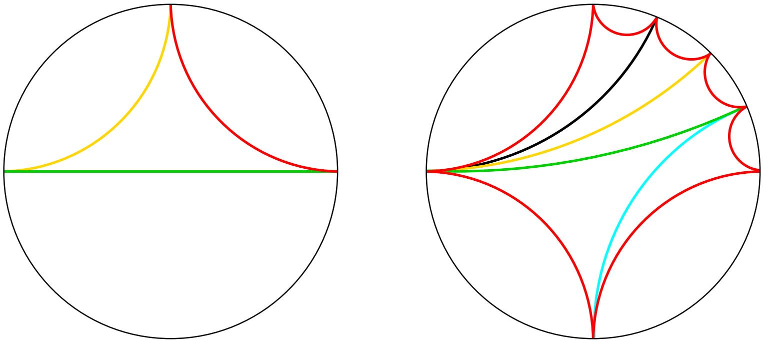


Transitivity of the action





Construction of $f \in \mathcal{T}$ such that $f \cdot E = v$.



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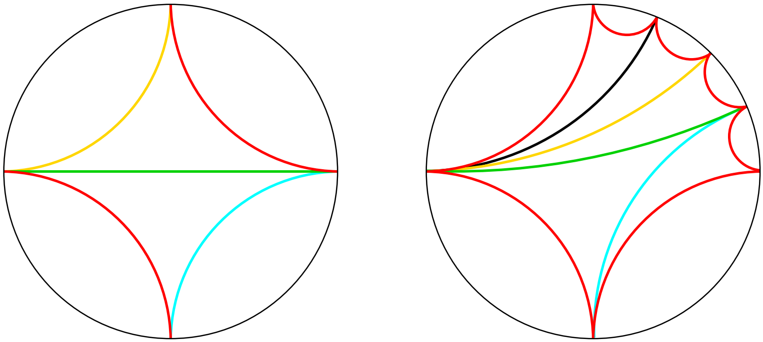

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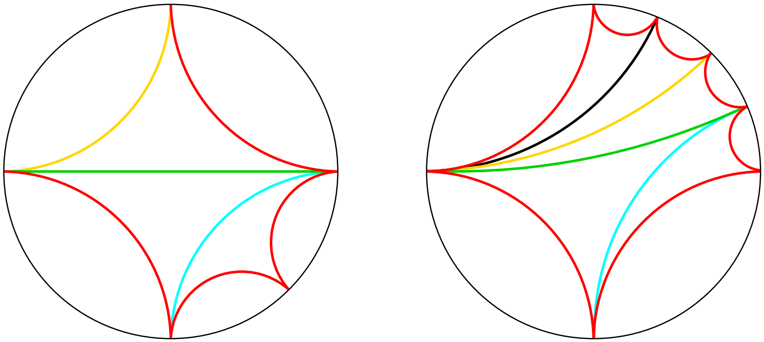

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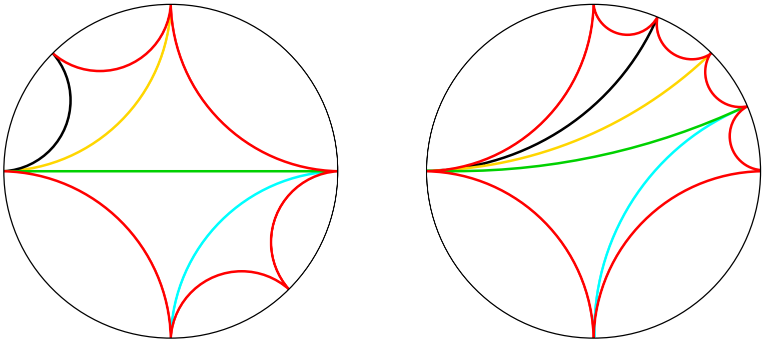

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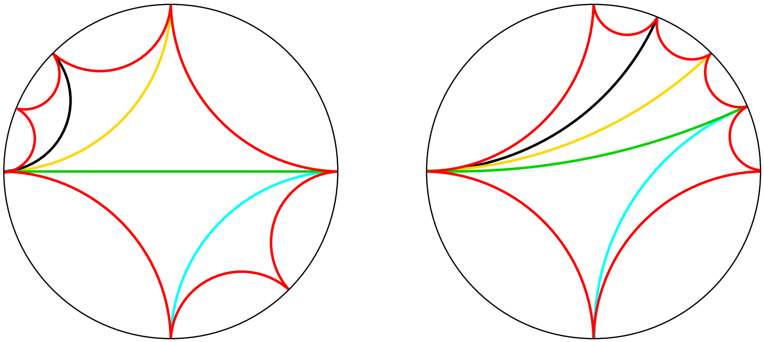

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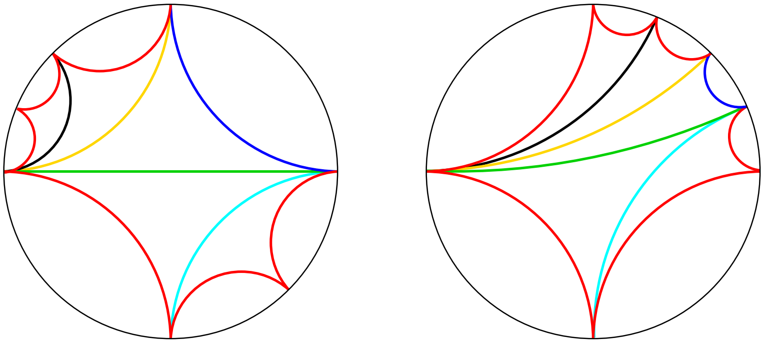

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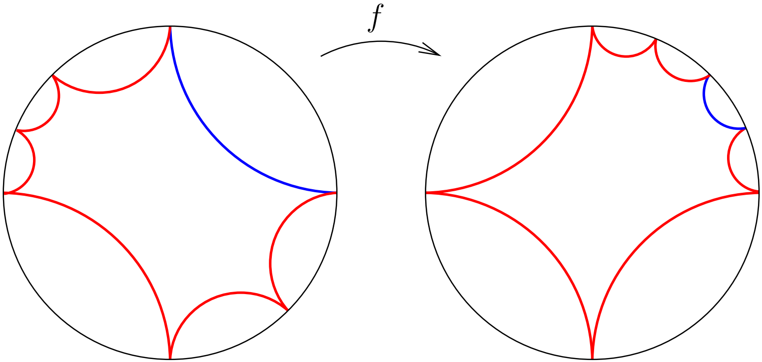
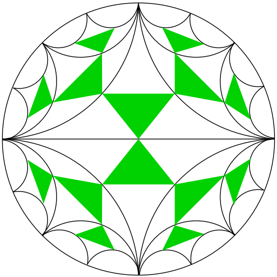


Link complex

Definition

Let $v \in \mathcal{C}^0$. The **link complex** $\mathcal{L}^2(v)$ of v is the simplicial complex

- Vertices: Neighbours of v in $\mathcal{C}$.
- Edges: u and w are joined if u, v, w lie in a pentagonal 2-cell.
- 2-simplex: Triangles in the 1-skeleton $\mathcal{L}^1(v)$.



Automorphisms of $\mathcal{C}$ induce link isomorphisms

Lemma

Let $v \in \mathcal{C}^0$. Then, every automorphism $\varphi \in$ Aut($\mathcal{C}$) induces an isomorphism $\varphi_{*,v} : \mathcal{L}^2(v) \to \mathcal{L}^2(w)$, where $w = \phi(v)$.

Question 1: what about the reciprocal?

Given v and w be vertices of $\mathcal{C}$ and $i : \mathcal{L}^2(v) \to \mathcal{L}^2(w)$ isomorphism, can we always find $\varphi \in$ Aut($\mathcal{C}$) such that $\varphi_{*,v} = i$?
NO.

Question 2: why not?

Which is the main obstruction to the existence of this automorphism?

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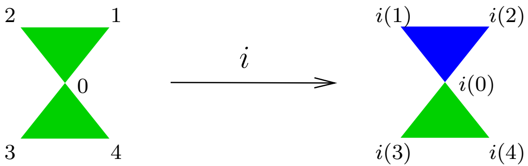
Main obstruction to extension of link isomorphisms

Proposition

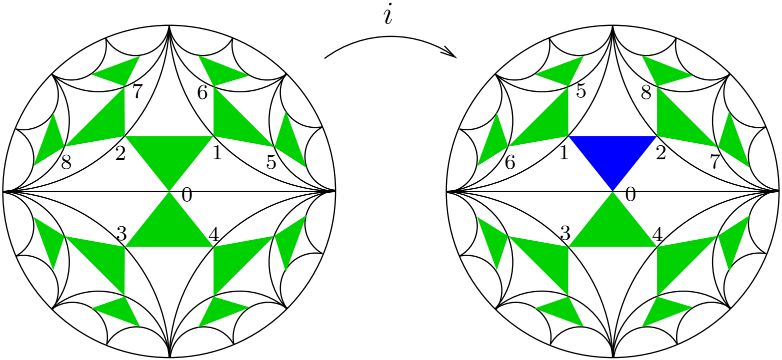
Let $i : \mathcal{L}^2(v) \to \mathcal{L}^2(w)$ be a link isomorphism such that:

- it is orientation reversing on $\Delta_1 = (u_0, u_1, u_2)$, and
- it is orientation preserving on $\Delta_2 = (u_0, u_3, u_4)$.

Then, there does not exist $\varphi \in$ Aut($\mathcal{C}$) with $\varphi_{*,v} = i$.



Example of non-extensible link isomorphism



Extension of link isomorphisms

Lemma

$v, w \in \mathcal{C}^0$, and $i : \mathcal{L}^2(v) \to \mathcal{L}^2(w)$ v-orientation preserving isomorphism. Then, there exists a unique automorphism $\varphi_i \in \text{Aut}(\mathcal{C})$ such that $\varphi_{*,v} = i$.
Furthermore, φ_i **is an element of** $\mathcal{T}$.

Remark

For all $\varphi \in \mathcal{T} \leq \text{Aut}(\mathcal{C})$ and for all v vertex of $\mathcal{C}$, $\varphi_{*,v}$ is v-orientation preserving.

Lemma

Let $i_R : \mathcal{L}^2E \to \mathcal{L}^2E$ be the orientation reversing isomorphism obtained by the symmetry of axis I_0^1. Then, there exists a unique automorphism $\varphi_R \in \text{Aut}(\mathcal{C})$ such that $\varphi_{R_{*,E}} = i_R$.

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Structure of Aut($\mathcal{C}$)

Theorem (F.-Nguyen, 2011)

$$\mathrm{Aut}(\mathcal{C}) \simeq \mathcal{T} \rtimes \mathbb{Z}/2\mathbb{Z}.$$

Proof. Consider

$$\Psi : \varphi \mapsto \begin{cases} 0, & \text{if } \varphi \text{ is orientation preserving} \\ 1, & \text{if } \varphi \text{ is orientation reversing.} \end{cases}$$

$\ker(\Psi) = \mathcal{T}$.
Section: $1 \in \mathbb{Z}/2\mathbb{Z} \dashrightarrow \varphi_R$.
Thus, we have

$$1 \longrightarrow \mathcal{T} \longrightarrow \mathrm{Aut}(\mathcal{C}) \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow 1,$$

and it splits.

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