

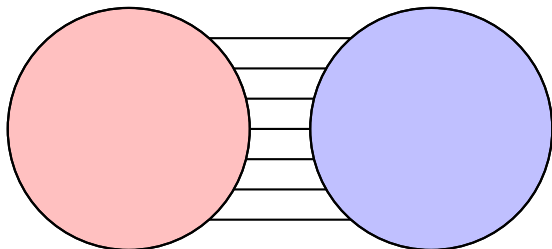
Almost Invariant Sets

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Introduction

Let G be a finitely generated group with finite generating set S and let $X = \text{Cay}(G, S)$ be the Cayley graph of G with respect to S . We say that G has more than one end if by removing finitely many edges of X we can disconnect the graph into more than one infinite component.

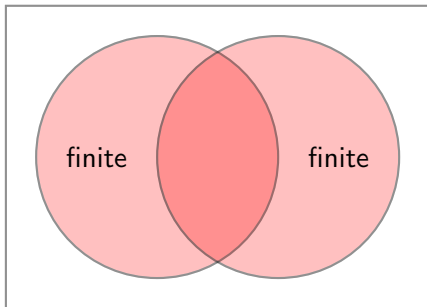


For example, the Cayley graph of a free group with respect to a free generating set S is a tree and removing just one edge gives two infinite components.

The property of having more than one end does not depend on the choice of generating set. The Cayley graph of a free group has more than one end regardless of whether one chooses a free generating set or not.

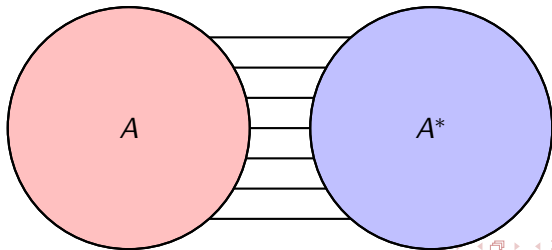
Let G, S be as above, and let $A \subset G$ be the vertices of one of the two infinite components obtained by removing finitely many edges of X . Then $A^* = G - A$ is the other component. Both A and A^* are infinite. Also A_s is almost equal to A for each $s \in S$. We write $A =^a A_s$ and say A almost equal to A_s if there are only finitely many elements of $A \cup A_s$ that are not in $A \cap A_s$

Almost equal sets



A subset $A \subset G$ is said to be **almost invariant** if $Ag \approx_a A$ for every $g \in G$. An almost invariant set is **proper** if both A and $A^* = G - A$ are infinite.

If S is a generating set for G , then A is almost invariant if and only if $As \approx_a A$ for every $s \in S$. We say that a group G has more than one end if G has a proper almost invariant subset. If G has a finite generating set, then this accords with the earlier geometric definition.



Theorem (Stallings 1971, Dicks-Dunwoody 1989)

A group G contains a proper almost invariant subset (i.e. it has more than one end) if and only if it has a non-trivial action on a tree with finite edge stabilizers.

The action of a group G on a tree is **trivial** if there is a vertex that is fixed by all of G . Every group has a trivial action on a tree.

Let T be a tree with directed edge set ET . If e is a directed edge, then let \bar{e} denote e with the reverse orientation. If e, f are distinct directed edges then write $e > f$ if the smallest subtree of T containing e and f is as below.



Suppose the group G acts on T . We say that g **shifts** e if either $e > ge$ or $ge > e$. If for some $e \in ET$ and some $g \in G$, g shifts e , then G acts non-trivially on a tree T_e obtained by contracting all edges of T not in the orbit of e or \bar{e} . In this action there is just one orbit of edge pairs. Bass-Serre theory tells us that either $G = G_u *_{G_e} G_v$ where u, v are the vertices of e and they are in different orbits in the contracted tree T_e or G is the HNN-group $G = G_u *_{G_e}$ if u, v are in the same G -orbit.

We say that G splits over G_e .

If there is no edge e that is shifted by any $g \in G$, (and G acts without involutions, i.e. there is no $g \in G$ such that $ge = \bar{e}$) then G must fix a vertex or an end of T . If the action is non-trivial it fixes an end of T , i.e. G is a union of an ascending sequence of vertex stabilizers, $G = \bigcup G_{v_n}$, where v_1, v_2, \dots is a sequence of adjacent vertices and $G_{v_1} \leq G_{v_2} \leq \dots$ and $G \neq G_{v_n}$ for any n .

Thus our theorem could be restated as

Theorem (Stallings 1971, Dicks-Dunwoody 1989)

A group G contains a proper almost invariant subset (i.e. it has more than one end) if and only if it splits over a finite subgroup or it is countably infinite and locally finite.

The if part of the theorem is fairly easy to prove. We show a stronger version of the if part.

Let H be a subgroup of G . A subset A is *H -finite* if A is contained in finitely many right H -cosets, i.e. for some finite set F , $A \subseteq HF$. A subgroup K is *H -finite* if and only if $H \cap K$ has finite index in K . Let T be a G -tree and suppose there is an edge e and vertex v .

We say that e **points at** v if there is a subtree of T as below. We write $e \rightarrow v$.

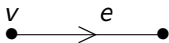


Let $G[e, v] = \{g \in G \mid e \rightarrow gv\}$.

If $h \in G$, then $G[e, v]h = G[e, h^{-1}v]$, since if $e \rightarrow gv$, $e \rightarrow gh(h^{-1}v)$.

If $K = G_v$, then $G[e, v]K = G[e, v]$. Also if $H = G_e$, then $HG(e, v) = G(e, v)$.

If $v = \iota e$, then $G_e = H \leq K = G_v$ and if $A = G[e, \iota e]$, then $A = HAK$.



Consider the set $Ax, x \in G$. If $g \in A, gx \notin A$, then $e \rightarrow gv, \bar{e} \rightarrow gxv$. This means that e is on the directed path joining gv and gxv . This happens if and only if $g^{-1}e$ is on the path joining v and xv . There are only finitely many directed edges in the G - orbit of e in this path. Hence $g^{-1} \in FH$, where F is finite, and $H = G_e$, and $g \in HF^{-1}$. Thus $A - Ax^{-1} = HF^{-1}$, i.e. $A - Ax^{-1}$ is H -finite. It follows that both $Ax - A$ and $A - Ax$ are H -finite and so $A + Ax$ is H -finite for every $x \in G$, i.e. A is an H -almost invariant set.

If the action on T is non-trivial, then neither A nor A^* is H -finite. We say that A is **proper**.

Conjecture (KC)

Let G be a group and let H be a subgroup. If there is a proper H -almost invariant subset A such that $A = AH$, then G has a non-trivial action on a tree in which every edge orbit contains an edge with an H -finite edge stabilizer.

For G finitely generated, this is a conjecture of Peter Kropholler (1990). We have seen that the conjecture is true if H has one element. The conjecture has been proved in many cases for H and G satisfying extra conditions by Kropholler, Dunwoody and Roller, Kleiner and M. Kapovich, Niblo, Kar and Niblo.

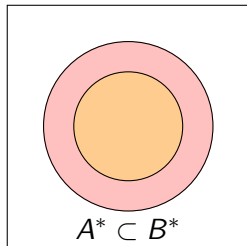
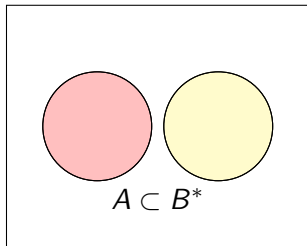
If G is the triangle group $G = \langle a, b \mid a^2 = b^3 = (ab)^7 = 1 \rangle$, then G has an infinite cyclic subgroup H for which there is a proper H -almost invariant set. Note that in this case G has no non-trivial action on a tree, so the condition $A = AH$ is necessary in KC.

Let U be a set. If $A \subset U$, then A^* denotes its complement in U .

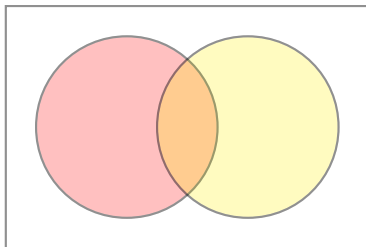
Two subsets A, B of U are **nested** if one of the four sets $A \cap B, A \cap B^*, A^* \cap B, A^* \cap B^*$ is empty.

Equivalently, A, B are nested if one of the inclusions $A \subset B, A \subset B^*, A^* \subset B, A^* \subset B^*$ holds.

Nested sets



Non-nested sets



A set \mathcal{E} of subsets of U is said to be **nested** if every pair $A, B \in \mathcal{E}$ is nested.

If A, B are almost invariant sets, then so are $gA, g \in G, A \cap B$ and $A + B$, the symmetric difference of A and B . Thus the set $\mathcal{B}G$ of almost invariant sets forms a $\mathbb{Z}_2 G$ -module which is also a Boolean algebra.

Theorem (Dicks and Dunwoody)

If G is a finitely generated group, $\mathcal{B}G$ has a nested G -set of generators.

Let T be a tree with directed edge set. If e is a directed edge, then let \bar{e} denote e with the reverse orientation. If e, f are distinct directed edges then write $e > f$ if the smallest subtree of T containing e and f is as below.



If e, f are distinct directed edges of a tree, then exactly one of $e > f, e > \bar{f}, \bar{e} > f, \bar{e} > \bar{f}$ holds.

Notice that if $e > f$, then there are only finitely many directed edges g such that $e > g > f$.

Theorem (MJD, 1979)

Let \mathcal{E} be a nested set of subsets of a set U , which is closed under taking complements. Suppose also that for every pair $A, B \in \mathcal{E}$ there is a *finite interval condition*, namely

if $A \supset B$, then there are only finitely many $C \in \mathcal{E}$ such that $A \supset C \supset B$.

Then there is a tree $T = T(\mathcal{E})$ for which \mathcal{E} is the set of directed edges of T and $A \supset B$ as subsets of U if and only if $A > B$ as directed edges of T .

The G -module $\mathcal{B}G$ has a nested G -set of generators which satisfies this finite interval condition if and only if the group is [accessible](#). In the action of G on the tree T each vertex stabilizer has at most one end. There are examples of groups which are not accessible (MJD 1991). For such a group the nested G -set $\mathcal{E}G$ of generators of $\mathcal{B}G$ does not satisfy the finite interval condition. However it forms the edge set of a generalized tree called a [protree](#). Any G -finite G -subset of \mathcal{E} does satisfy the finite interval condition and so is the edge set of a G -tree.

Muller and Schupp (1983) showed that a group G is context-free if and only if it has a Cayley graph Γ in which every closed path in Γ is m -triangulable. They also showed that if G is accessible and m -triangulable, then G is virtually free. But m -triangulable implies finitely presented and finitely presented groups are accessible (MJD 1985) and so group is context free if and only if it is virtually free.

Recently Yago Antolin has used an earlier 1979 result of mine to give a new proof that a finitely generated group G is virtually free if and only if every closed path in a Cayley graph Γ is m -triangulable.

Thus

G context-free

\iff every closed path in a Cayley graph Γ is m -triangulable

$\iff G$ virtually free.

The Kropholler Conjecture

Let G be a finitely generated group and let $H \leq G$. Let $A \subset G$ be such that $AH = A$ and also $Ag + A$ is H -finite for every $g \in G$, and also neither A nor $A^* = G - A$ is H -finite. Then KC says there is a G -tree in which every edge orbit contains an edge with an H -finite edge stabilizer and no vertex is fixed by G .

Kropholler showed that by replacing H by a possibly smaller subgroup, and replacing A by a possibly different H -almost invariant set, we can assume that $A = HAK$ where $H \leq K$.

A new approach to KC is now outlined. The idea is to give a new proof of the case for $H = \langle 1 \rangle$, which is known to be true, that generalizes to all subgroups H .

Let $X = \{B \mid B =_a A\}$ so that for $B, C \in X, B + C = HF$ where F is finite. For $H = \{1\}$ it follows from the Almost Stability Theorem (Dicks and MJD 1989) that X is the vertex set of a G -tree. This is proved by getting a stronger form of Stallings Theorem for finitely generated groups and using an transfinite induction argument for the general case.

There is a simpler proof (I think) for general H that uses the only recently noticed structure on X as a metric space.

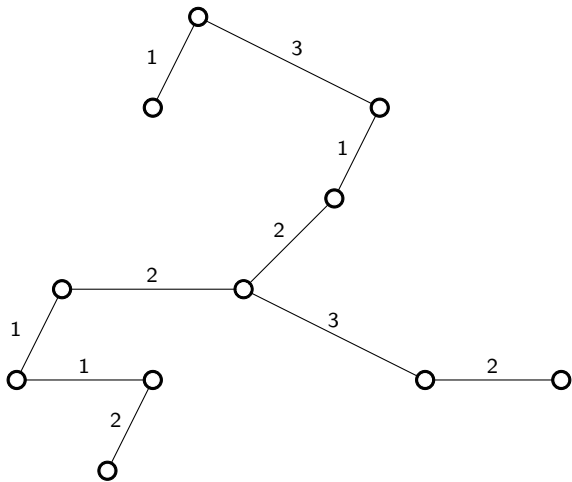
For $B, C \in X$ define $d(B, C)$ to be the number of cosets in $B + C$.

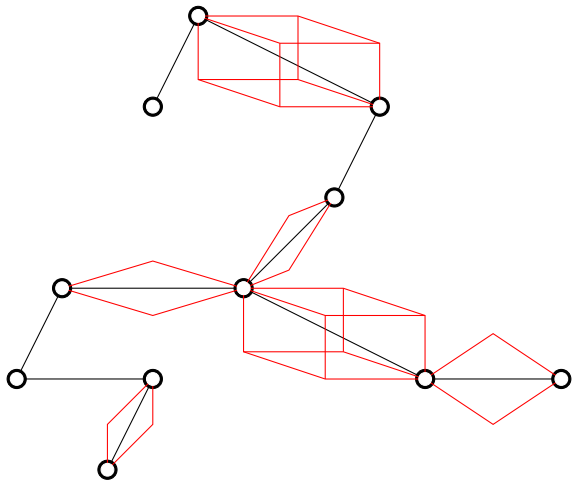
This is a metric on X since $(B + C) + (C + D) = (B + D)$, and so an element which is in $B + D$ is in just one of $B + C$ or $C + D$.

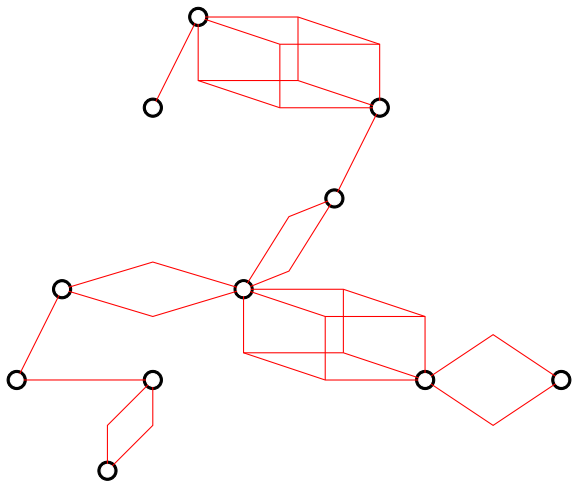
Also G acts on X by isometries, since $(B + C)z = Bz + Cz$.

Theorem (H -AST)

The metric G -space X has a subspace that is the vertex set of reduced G -tree T in which each edge orbit contains an edge with an H -finite edge stabilizer.







One could think of H -AST as being the H -Almost Stability Theorem.

H -AST \implies KC

A proof ? of H -AST

Form the graph Γ in which $V\Gamma = X$ and two vertices are joined by an edge if they are distance one apart. Then Γ is a connected G -graph. Every edge in Γ corresponds to a particular H -coset. There are $n!$ geodesics joining B and C if $d(B, C) = n$.

The edges corresponding to a particular coset Hb disconnect Γ .

Removing this set of edges gives two sets of vertices, B and B^* , where B is the set of those $C \in X$ such that $Hb \subset C$.

Suppose that G is finitely generated by the set $\{g_1, g_2, \dots, g_n\}$. Let Γ' be the G -subgraph of Γ consisting of all points that lie on a geodesic joining points A and Ag_i for every $i = 1, 2, \dots, n$ and every $g \in G$. This graph is connected.

There are only finitely many edges corresponding to a particular H -coset in Γ' .

There are $n!$ geodesics joining A and Ag_i if $d(A, Ag_i) = n$. Hence there are only finitely many cosets on these geodesics. There are only finitely many translates of these geodesics that will also contain one of these cosets.

We have a connected G -graph Γ' that can be disconnected by removing finitely many edges.

The theory of structure trees of Dunwoody 1982 and Dicks and Dunwoody, Chapter II applies in this situation. We get a G -tree T for which there is an injective G -map $\theta' : V\Gamma' \rightarrow VT$.

The general case could follow from the finitely generated case fairly easily using the metric space structure.

References

Yago Antolin [On Cayley graphs of virtually free groups](#) (2012)
arXiv:0911.2177.

Warren Dicks, [Group, trees and projective modules](#) Springer
Lecture Notes [790](#) 1980

Warren Dicks and M.J.Dunwoody, [Groups acting on graphs](#),
Cambridge University Press, 1989.

M.J.Dunwoody, [The accessibility of finitely presented groups](#),
Invent. Math. [81](#) (1985) 449-457.

M.J.Dunwoody, [Cutting up graphs](#), *Combinatorica* [1](#) (1982) 15-23.
449-457.

M.J.Dunwoody, [Accessibility and groups of cohomological dimension one](#), Proc. London Math. Soc. (3) **38** (1979) 195-215.

M.J.Dunwoody and M.Roller, [Splitting groups over polycyclic-by-finite subgroups](#), Bull. London Math.Soc. **23** 29-36 (1989).

M.J.Dunwoody, [An inaccessible group](#), London Math. Soc. Lecture Note Series **181** 173-78 (1991).

P.H.Kropholler, [An analogue of the torus decomposition theorem for certain Poincaré groups](#), Proc. London Math. Soc. (3) **60** 503-529 (1990)

P.H.Kropholler, [A group theoretic proof of the torus theorem](#), London Math. Soc. Lecture Note Series **181** 138-158 (1991).

G.A. Niblo, [A geometric proof of Stallings theorem on groups with more than one end](#), Geometriae Dedicata **105**, 61-76 (2004).

J.R.Stallings, [Group theory and three-dimensional manifolds](#), Yale Math. Monographs **4** (1970).