Geodesically Perfect Rewriting Systems

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joint work with

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Outline

1. Rewriting Systems
2. Finding nice rewriting systems
3. Infinite rewriting systems
4. Geodesic rewriting systems
5. Examples

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Rewriting Relations

A *rewriting relation* over a set X is a subset $\Longrightarrow \subseteq X \times X$.

- $\overset{*}{\Longrightarrow}$: the reflexive and transitive closure of $\Longrightarrow$;
- $\overset{*}{\Longleftrightarrow}$: its symmetric, reflexive, and transitive closure.

The relation $\Longrightarrow \subseteq X \times X$ is called:

- *confluent*, if $y \overset{*}{\Longleftarrow} x \overset{*}{\Longrightarrow} z$ implies $y \overset{*}{\Longrightarrow} w \overset{*}{\Longleftarrow} z$ for some w;
- *terminating*, if every infinite chain

$$x_0 \overset{*}{\Longrightarrow} x_1 \overset{*}{\Longrightarrow} \cdots x_{i-1} \overset{*}{\Longrightarrow} x_i \overset{*}{\Longrightarrow} \cdots$$

  becomes stationary;
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String rewriting system for a monoid

Γ a finite alphabet; Γ^* the free monoid Γ.

A *rewriting system* over Γ^* is a relation $S \subseteq \Gamma^* \times \Gamma^*$.

It defines the rewriting relation $\underset{S}{\Longrightarrow} \subseteq \Gamma^* \times \Gamma^*$ by

$$x \underset{S}{\Longrightarrow} y, \text{ if } x = p\ell q, \; y = prq \text{ for some } (\ell, r) \in S.$$

The relation $\overset{*}{\underset{S}{\Longleftrightarrow}} \subseteq \Gamma^* \times \Gamma^*$ is a congruence;

write Γ^*_S for the quotient monoid.

A *Thue system* is a rewriting system S for which, for all $(l, r) \in S$ we have $|l| \geq |r|$ and if $|l| = |r|$ then $(r, l) \in S$ as well.

We may always assume S is Thue.(If $(l, r) \in S$ then adding (r, l) does not change the congruence.)

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Cyclic words

A *cyclic word* of *length* $n \geq 0$ over an alphabet Γ is a directed cycle graph of length n, each edge of which is labelled by an element of Γ.

The set of all cyclic words over Γ is denoted $\Gamma^{\circledast}$.

If $p = u_0, e_1, \ldots, e_d, u_d$ is a directed path of length d in a cycle graph and e_i has label a_i then the *label* $l(p)$ of p is the word $a_1 \cdots a_n \in \Gamma^*$.

If C is a cycle of length n and p is a path in C of length $d \leq n$ then we say that p is a *factor* of C.

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Equality of cyclic words

Let C be a cyclic word of length n. If a vertex α (called a *base-point*) is chosen then there is a unique path p_α of length n in C which starts and ends at α.

Denote by C_α the label $l(p_\alpha)$ of p_α.

Say that two cyclic words C and D are equal if there are base-points $\alpha \in C$ and $\beta \in D$ such that $C_\alpha = D_\beta$.

Conversely, given a word $u \in \Gamma^*$ one can form a cyclic word u°, the cyclic word *represented* by u, which is defined uniquely by the condition that $u^\circ_\alpha = u$ for some based-point $\alpha \in u^\circ$.

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Cyclic rewriting system

A rewriting systm S over the free monoid Γ^* induces the *cyclic rewriting relation* $\underset{S}{\circ\!\!\Longrightarrow} \subseteq \Gamma^{\circledast} \times \Gamma^{\circledast}$ such that $C \underset{S}{\circ\!\!\Longrightarrow} D$ if and only if either

(i) there is a factor p of C with the label ℓ and a rule $\ell \to r$ in S such that D is obtained from C by replacing p with a path q with label r and with the same initial and terminal vertices as p; or

(ii) C and D have length 1 and labels a and $b \in \Gamma$, respectively, and S contains rules

$$ts \longrightarrow a$$

and

$$st \longrightarrow b$$

for some $s, t \in \Gamma^*$.

(Or one of a finite list of other similar rules.)

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Cyclic presentations

As usual write
$D \underset{S}{\Leftarrow\!\circ} C$ if $C \underset{S}{\circ\!\Rightarrow} D$,
$\overset{*}{\underset{S}{\circ\!\Rightarrow}}$ for the reflexive, transitive closure of $\underset{S}{\circ\!\Rightarrow}$,
$\overset{*}{\underset{S}{\Leftarrow\!\circ\!\Rightarrow}}$ for its reflexive, symmetric, transitive closure and
$\overset{*}{\underset{S}{\Leftarrow\!\circ}}$ for the transpose of $\overset{*}{\underset{S}{\circ\!\Rightarrow}}$.

If P is a property of rewriting systems (Church-Rosser, strongly confluent, confluent, etc.) we say that a monoid M has a cyclically P-presentation if it can be defined by a system for which the relation $\underset{S}{\circ\!\Rightarrow}$ has property P.

Transposition and conjugacy

Let $M = \Gamma^*/S$ be a monoid. Consider the *S-transposition* relation $\sim$ on Γ^* given by $u \sim v$ if and only if there exist elments $r, s \in \Gamma^*$ such that $u =_M rs$ and $v =_M sr$. The transitive closure of $\sim$ is an equivalence relation on Γ^*.

Let $\Gamma = \Sigma \cup \Sigma^{-1}$ and $S \subseteq \Gamma^* \times \Gamma^*$ be a rewriting system such that $G_S = \Gamma^*/S$ is a group. For $u, v \in \Gamma^*$ we write $u \sim_S v$ if the elements defined by the words u and v in G_S are conjugate in G_S.

Lemma

Let S be a rewriting system over Γ^*, let $C, D \in \Gamma^{\circledast}$ and let α and β be base-points of C and D, respectively.

(i) *If $C \underset{S}{\circ\!\!\Longrightarrow} D$ then C_α and C_β are transposed in Γ^*/S.*

(ii) *If $C \underset{S}{\overset{*}{\Longleftarrow\!\circ\!\Longrightarrow}} D$ then C_α and C_β are conjugate in Γ^*/S (if it's a group).*

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Lemma

Let S be a rewriting system over Γ^* and let $\sim_S$ be the transitive closure of the S-transposition relation on Γ^*. Let $u, v \in \Gamma^*$ and let u° and v° be cyclic words represented by u and v, respectively. Then $u \sim_S v$ if and only if $u^\circ \overset{*}{\underset{S}{\Longleftrightarrow}} v^\circ$.

Corollary

Let S be a rewriting system over Γ^*, let $\sim_S$ be the transitive closure of the S-transposition relation on Γ^* and let $u \in \Gamma^*$. Then the cyclic word u° is S-cyclically geodesic if and only if u is of minimal length in its $\sim_S$ class in G_S.

Corollary

Let $\Gamma = \Sigma \cup \Sigma^{-1}$ and $S \subseteq \Gamma^* \times \Gamma^*$ be a rewriting system such that $G_S = \Gamma^*/S$ is a group. Let $u, v \in \Gamma^*$. Then u and v are conjugate in G_S if and only if $u^\circ \overset{*}{\underset{S}{\Longleftrightarrow}} v^\circ$. Moreover u is a minimal length representative of its conjugacy class if and only if u° is S-cyclically geodesic.

Finding rewriting systems

A rewriting system (with some specific desirable properties) can sometimes be constructed using a class of algorithms known as Knuth-Bendix (KB).

- start with a naive system, for example, given a presentation $\langle X \mid R \rangle$ rewrite relations as $\ell_i = r_i$, $|\ell_i| \geq |r_i|$, and form an associated system of rewrite rules $\ell_i \to r_i$.
- apply a **Knuth-Bendix (KB) completion process**:
  - if there's a pair of rules violating confluence (a critical pair) then resolve the ambiguity by adding a new rule;
  - remove redundant rules;
  - repeat.
- If the process halts the output is a confluent rewriting system.

KB algorithms are sensitive to small adjustments in configuration of the input.

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- start with a naive system, for example, given a presentation $\langle X \mid R \rangle$ rewrite relations as $\ell_i = r_i$, $|\ell_i| \geq |r_i|$, and form an associated system of rewrite rules $\ell_i \to r_i$.
- apply a **Knuth-Bendix (KB) completion process**:
  - if there's a pair of rules violating confluence (a critical pair) then resolve the ambiguity by adding a new rule;
  - remove redundant rules;
  - repeat.
- If the process halts the output is a confluent rewriting system.

KB algorithms are sensitive to small adjustments in configuration of the input.

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Finite convergent rewriting systems

A word is S-**geodesic** if it's of minimal length in its $\overset{*}{\underset{S}{\Longleftrightarrow}}$ equivalence class and S-**cyclically geodesic** if it's of minimal length in its $\overset{*}{\underset{S}{\Leftarrow\!\circ\!\Rightarrow}}$ equivalence class.

Suppose $S \subseteq \Gamma^* \times \Gamma^*$ is a finite convergent rewriting system. Then

- S-geodesic representatives of elements are unique and may be effectively computed(every $\overset{*}{\underset{S}{\Longleftrightarrow}}$ equivalence class contains a unique S-reduced word);
- the monoid Γ^*_S has solvable word problem; and
- if there exists a finite convergent rewriting system $T \subseteq \Gamma^* \times \Gamma^*$ then a Knuth-Bendix procedure applied to S will output one.

e.g. finite groups, polycyclic groups, free groups, surface groups, ... but ...

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Disadvantages of FCRWS

- Algorithms for the WP and geodesics may be of arbitrarily (unnecessarily) high complexity (Otto and Kobayashi); and
- there is no algorithm to determine whether or not a finite Thue system is convergent (Ó'Dúnlaing).

Moreover the KB process depends on finding an appropriate reduction ordering on Γ^*.

For example, in the free Abelian group of rank 2 the KB procedure outputs a finite convergent rewriting system if the generators are ordered $x < x^{-1} < y < y^{-1}$.

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Open Problems

Do there exist finite convergent presentations for

- hyperbolic groups?
- automatic groups?
- one-relator groups?
- fully residually free groups?

Conjecture (R Gilman)

Let G be a finitely generated group. Then the following assertions are equivalent:

1) *G is a plain group, i.e., G is a free product of free and finite groups.*
2) *G has a finite convergent length-reducing presentation.*
3) *G has a finite convergent weight-reducing presentation.*

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Finite convergent cyclic rewriting

Chouraqui defines a relation $\looparrowright$ on Γ^*. The equivalence relation generated by $\looparrowright$ relates two words u and v in Γ^* if and only if $u^\circ \overset{*}{\underset{S}{\Longleftrightarrow}} v^\circ$.

Fabienne Chouraqui

"The Knuth-Bendix algorithm and the conjugacy problem in monoids", Semigroup Forum, No. 1, **82**, 181–196, (2011) Springer.

Chouraqui finds conditions under which the reflexive and transitive closure of $\looparrowright$ is convergent. This gives rise to algorithms for the transposition and conjugacy problems in certain monoids.

For us the focus is on rewriting systems which may be infinite and do not necessarily terminate.

Computing with infinite systems

To use an infinite string rewriting system $S \subseteq \Gamma^* \times \Gamma^*$ it's necessary to be able to

- recognize if a given pair $(u, v) \in \Gamma^* \times \Gamma^*$ is a rule of S or not, (S must be a recursive subset of $\Gamma^* \times \Gamma^*$) and
- check, given $u \in \Gamma^*$, whether or not there is a rule $\ell \to r \in S$ with $\ell = u$: so we assume that the set $L(S)$ of the left-hand sides of the rules in S is a recursive subset of Γ^*.

Systems satisfying these two conditions are termed *effective* rewriting systems.

Proposition (Diekert, Duncan, Miasnikov (D,D,M))

Let S be an infinite effective system. If S is convergent then the word problem in the monoid $M = \Gamma^*_S$ defined by S is decidable. If S is cyclically convergent and M is a group then its conjugacy problem is also decidable.

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Geodesic rewriting systems

Definition

A string rewriting system $S \subseteq \Gamma^* \times \Gamma^*$ is called *(cyclically)geodesic* if S-(cyclically)geodesic words are exactly those words to which no length reducing rule can be applied.

Theorem (D,D,M)

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Geodesically perfect rewriting systems

Definition

A Thue system $S \subseteq \Gamma^* \times \Gamma^*$ is called *geodesically perfect*, if S is

i) geodesic, and

ii) if $u, v \in \Gamma^*$ are S-geodesics, then $u \overset{*}{\underset{S}{\Longleftrightarrow}} v$ if and only if $u \overset{*}{\underset{S_P}{\Longleftrightarrow}} v$, where S_P is the length-preserving part of S.

Theorem (Nivat and Benois)

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Theorem (D,D,M)

There is a Knuth-Bendix procedure that converts a finite Thue system S to an equivalent geodesically perfect Thue system T. The procedure halts iff the monoid defined by S can be defined by a finite geodesically perfect system: in which case the output T is finite.

Proposition

Let S be a geodesically perfect Ptime enumerable string rewriting system such that the monoid $G = \Gamma_S^*$ is a group. Then the word problem in G is decidable in polynomial time.

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Cyclically geodesically perfect rewriting systems

Definition

A rewriting system $S \subseteq \Gamma^* \times \Gamma^*$ is called *cyclically geodesically perfect*, if

i) S is cyclically geodesic, and

ii) if $u, v \in \Gamma^{\circledast}$ are S-cyclic geodesics then $u \overset{*}{\underset{S}{\Longleftrightarrow}} v$ if and only if $u \overset{*}{\underset{S}{\circ\!\!=\!\!\circ}} v$, where $\underset{S}{\circ\!\!=\!\!\circ}$ is the length preserving part of $\underset{S}{\Longleftrightarrow}$.

Theorem (D,D,M)

If S is effective and cyclically geodesically perfect then G has solvable conjugacy problem.

Examples

- partially commutative groups (RAAGs, graph groups) and monoids: finite preperfect and geodesically perfect systems, respectively. WP is linear in both (Book and Otto '93).
- Coxeter groups: finite preperfect systems.
- Free products with amalgamation: natural (infinte) geodesically perfect and cyclically geodesically perfect systems.
- HNN extensions: natural (infinite) convergent systems & (infinite) geodesically perfect and cyclically geodesically perfect systems.

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Free products with amalgamation

Let A and B be groups with $A \cap B = H$ and let $G = A *_H B$.

Let $\Gamma = A \cup B$ (with elements of H identified) and
if $u, v \in A$ or $u, v \in B$ write $[uv]$ for the product in A or B respectively (so $[uv]$ is viewed as a letter in Γ).

Write ε for the identity element of H (and A and B) and 1 for the empty word.

The system $S \subseteq \Gamma^2 \times (\{1\} \cup \Gamma)$ is defined by the rules:

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\begin{array}{rcll}
\varepsilon & \longrightarrow & 1 & \\
ab & \longrightarrow & [ab] & \text{if } [ab] \text{ is defined,} \\
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The word problem for G is dependent (only) on the word problems for A and B and the membership problem for H.
We obtain short proofs of the "conjugacy criterion" for free products with amalgamation (cf. Magnus, Karrass, Solitar).

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