Intersection problem for free times free-abelian groups

Jordi Delgado
(joint with Enric Ventura)

Universitat Politècnica de Catalunya

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Índex

Free times free-abelian f. g. groups

Let $n, m \in \mathbb{N}$, and $X = \{x_1 \ldots, x_n\}$, $T = \{t_1, \ldots, t_m\}$.
We consider the group

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We have the ***normal form***:

$$t_1^{a_1} \cdots t_m^{a_m} u(x_1, \ldots, x_n) =: \mathbf{t}^{\mathbf{a}} u \ ,$$

where $\mathbf{a} = (a_1, \ldots, a_m) \in \mathbb{Z}^m$ and $u \in F_n$.

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Remarks

- $(F_n \times \mathbb{Z}^m)^{\text{ab}} \cong \mathbb{Z}^{n+m}$
- $F_1 \times \mathbb{Z}^m \cong F_0 \times \mathbb{Z}^{m+1}$
- $F_n \times \mathbb{Z}^m$ is abelian (and free-abelian) $\Leftrightarrow$ $n = 0, 1$.

Subgroups

$G = F_n \times \mathbb{Z}^m$

Proposition

For any $H \leqslant G$ we have

$$H = H\pi\alpha \times (H \cap \mathbb{Z}^m)$$

where

- $H\pi\alpha \cong H\pi$ is free of rank $n' \in [0, \infty]$, and
- $H \cap \mathbb{Z}^m = \ker(\pi_{|H})$ is free-abelian of rank $m' \in [0, m]$.

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2. H f.g. $\Leftrightarrow$ $H\pi$ f.g. (where $\pi \colon \mathbf{t^a}\, u \mapsto u$),
3. If $n, n' \in \mathbb{N} \setminus \{1\}$ and $m, m' \in \mathbb{N}$, then

$$F_n \times \mathbb{Z}^m \cong F_{n'} \times \mathbb{Z}^{m'} \;\Leftrightarrow\; n = n' \text{ i } m = m'.$$

Basis

$H = H\pi\alpha \times (H \cap \mathbb{Z}^m)$

Definition

Given $H \leqslant G$, we say that $E \subseteq G$ is a ***basis*** of H if

$$E = E_X \sqcup E_T$$

where

- E_X basis of $H\pi\alpha$, and
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Proposition

If H is given by a finite family of generators, then we can compute a basis of H.

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$$E = \{\, \mathbf{t}^{\mathbf{a_1}} u_1, \ldots, \mathbf{t}^{\mathbf{a_{n'}}} u_{n'}, \mathbf{t}^{\mathbf{b_1}}, \ldots, \mathbf{t}^{\mathbf{b_{m'}}} \,\},$$

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Abelian completion (of a free word in a subgroup)

Definition

Given $H \leqslant G$ and $w \in G\pi = \langle X \rangle_G$,

$$\mathfrak{c}_{w,H} = \{ \mathbf{a} \in \mathbb{Z}^m \mid t^{\mathbf{a}} w \in H \} \subseteq \mathbb{Z}^m$$

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Lemma

- *if* $w \notin H\pi$, *then* $\mathfrak{c}_{w,H} = \emptyset$, *i*
- *if* $w \in H\pi$, *then* $\mathfrak{c}_{w,H} = \boldsymbol{\omega}\mathbf{A} + L$,

where $\boldsymbol{\omega}$ *is the abelianization of the word* ω *which expresses* w *in the basis* $\{u_1, \ldots, u_{n'}\}$ *of* $H\pi$, *i.e.* $w = \omega(u_1, \ldots, u_{n'})$.

Índex

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Example

Let $H = \langle ta, b \rangle$, $K = \langle t^{-1}a, b \rangle \leqslant F_2 \times \mathbb{Z} = \langle a, b \mid \rangle \times \langle t \mid \rangle$.
Then, $H \cap K = \langle \{ a^{-k} b a^k \mid k \in \mathbb{Z} \} \rangle$ is **not** finitely generated.

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So, what about the Howson property for free groups? ...

Subgroup intersection

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Proposition

If $\omega_1(\vec{u})$, $\omega_2(\vec{u}')$ are the expressions of $w \in H_1\pi \cap H_2\pi$ in basis $\vec{u}$ and $\vec{u}'$, and $\boldsymbol{\omega_1} \in \mathbb{Z}^{n_1}, \boldsymbol{\omega_2} \in \mathbb{Z}^{n_2}$ the abelianzations of ω_1 i ω_2, then

- $$H_1 \cap H_2 = \left\{ \mathbf{t}^{\mathbf{a}}\, w \in G \;\middle|\; \begin{array}{l} w = \omega_1(\vec{u}) = \omega_2(\vec{u}') \in H_1\pi \cap H_2\pi \\ \mathbf{a} \in (\boldsymbol{\omega_1}\mathbf{A_1} + L_1) \cap (\boldsymbol{\omega_2}\mathbf{A_2} + L_2) \end{array} \right\}$$
- $$(H_1 \cap H_2)\pi = \left\{ w \in F_n \;\middle|\; \begin{array}{l} w = \omega_1(\vec{u}) = \omega_2(\vec{u}') \in H_1\pi \cap H_2\pi \\ (\boldsymbol{\omega_1}\mathbf{A_1} + L_1) \cap (\boldsymbol{\omega_2}\mathbf{A_2} + L_2) \neq \emptyset \end{array} \right\}$$

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Remark

$$(H_1 \cap H_2)\pi \leqslant H_1\pi \cap H_2\pi \leqslant_{\text{f.g.}} F_n$$

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Remark

$$(H_1 \cap H_2)\pi < H_1\pi \cap H_2\pi \leqslant_{\text{f.g.}} F_n$$

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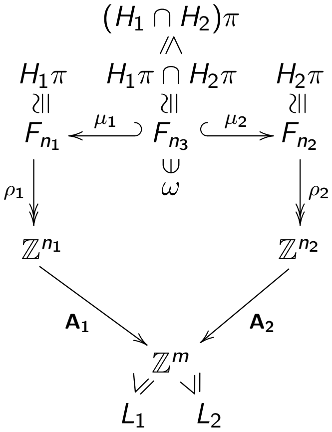

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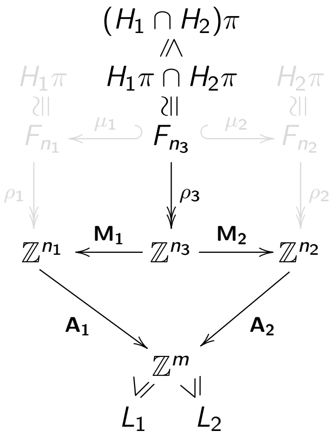

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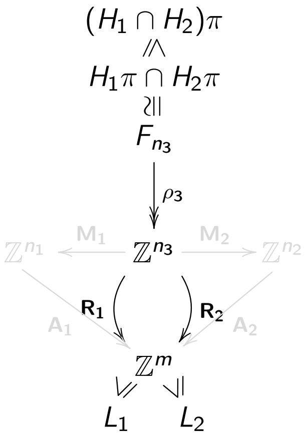

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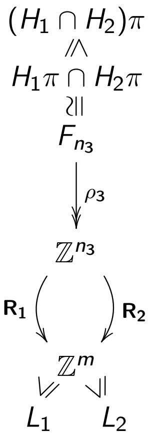
- $(\omega\rho_3\mathbf{R_1} + L_1) \cap (\omega\rho_3\mathbf{R_2} + L_2) \neq \emptyset$



$(H_1 \cap H_2)\pi$

All the $\omega(\vec{v}) \in H_1\pi \cap H_2\pi$ such that, equivalently:

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i.e., we have

Proposition

$$(H_1 \cap H_2)\pi \cong M\rho_3^{-1},$$

where $M = (L_1 + L_2)(\mathbf{R_1} - \mathbf{R_2})^{-1}$.

$$\begin{array}{c} (H_1 \cap H_2)\pi \\ \not\leq \\ H_1\pi \cap H_2\pi \\ \wr\Vert \\ F_{n_3} \\ \Big\downarrow \rho_3 \\ \mathbb{Z}^{n_3} \\ \Big\downarrow \mathbf{R_1} - \mathbf{R_2} \\ \mathbb{Z}^m \end{array}$$

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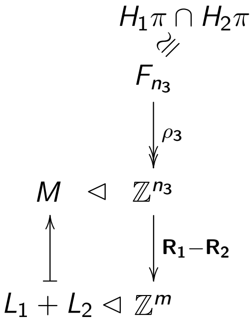

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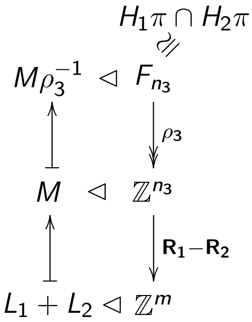

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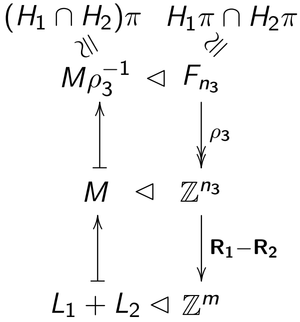

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Finite type of the intersection of two f. g. subgroups

Proposition

Let $H_1, H_2 \leqslant_{\text{f.g.}} G$,

- *if* $(H_1 \cap H_2)\pi \neq 1$, *then*

$$H_1 \cap H_2 \ \text{f. g.} \ \Leftrightarrow \ \text{rank}(M) = n_3,$$

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Proof

$$H_1 \cap H_2 \ \text{f. g.} \quad \Leftrightarrow \quad (H_1 \cap H_2)\pi \ \text{f. g.}$$

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Proof

$$\begin{aligned} H_1 \cap H_2 \ \text{f. g.} \quad &\Leftrightarrow \quad (H_1 \cap H_2)\pi \ \text{f. g.} \\ &\Leftrightarrow \quad M\rho_3^{-1} \ \text{f. g.} \end{aligned}$$

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Proof

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Proof

$$\begin{aligned} H_1 \cap H_2 \text{ f. g.} &\Leftrightarrow (H_1 \cap H_2)\pi \text{ f. g.} \\ &\Leftrightarrow M\rho_3^{-1} \text{ f. g.} \\ &\Leftrightarrow M\rho_3^{-1} \leqslant_{\text{f.i.}} F_{n_3} \\ &\Leftrightarrow M \leqslant_{\text{f.i.}} \mathbb{Z}^{n_3} \quad \Leftrightarrow \quad \operatorname{rank}(M) = n_3. \end{aligned}$$

Finite type of the intersection of two f. g. subgroups

Proposition

Let $H_1, H_2 \leqslant_{\text{f.g.}} G$,

- *if* $(H_1 \cap H_2)\pi \neq 1$, *then*

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Proof

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Finite type decision

Theorem

If $H_1, H_2 \leqslant G$ are given by finite families of generators, then

1. ***we can algorithmically decide*** *whether $H_1 \cap H_2$ is $\mathrm{f.g.}$, and*
2. *in afirmative case,* ***we can compute a basis*** *of $H_1 \cap H_2$.*

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If $H_1, H_2 \leqslant G$ are given by finite families of generators, then

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   3. *compute μ_1, μ_2, $\mathbf{M}_1$, $\mathbf{M}_2$, $\mathbf{R}_1 = \mathbf{M}_1\mathbf{A}_1$ and $\mathbf{R}_2 = \mathbf{M}_2\mathbf{A}_2$,*
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Computing a basis

Theorem (cont.)

2. in afirmative case, **we can compute a basis** of $H_1 \cap H_2$.
    1. Compute a basis $\{\mathbf{b_1}, \ldots, \mathbf{b_{m'}}\}$ of $L_1 \cap L_2$,
    2. if $(H_1 \cap H_2)\pi = 1$, $\{\mathbf{t}^{\mathbf{b_1}}, \ldots, \mathbf{t}^{\mathbf{b_{m'}}}\}$ is a basis of $H_1 \cap H_2$ (if not, continue).
    3. compute a s.c.r. S of M in $\mathbb{Z}^{n_3}$,
    4. compute $S\rho_3^{-1}$, a s.c.r. (right) of $M\rho_3^{-1}$ in F_{n_3},
    5. consider an empty graph with vertices the cosets in $S\rho_3^{-1}$,
    6. $\forall v_k$ generator of $H_1\pi \cap H_2\pi$, and $\forall z_l$ vertex of the graph, search $(M\rho_3^{-1})\, z_l v_k$ and add an edge with label v_k from $(M\rho_3^{-1})\, z_l$ to $(M\rho_3^{-1})\, z_l v_k$ (we obtain the Schreier graph for $M\rho_3^{-1}$ f. i. F_{n_3}),
    7. compute a basis of $M\rho_3^{-1}$ from a maximal tree,
    8. compute a basis $\{\mathbf{t}^{\mathbf{a_1}} v_1, \ldots, \mathbf{t}^{\mathbf{a_{n'}}} v_{n'}\}$ of $(H_1 \cap H_2)\pi\alpha$.
    9. $\{\mathbf{t}^{\mathbf{a_1}} v_1, \ldots, \mathbf{t}^{\mathbf{a_{n'}}} v_{n'}, \mathbf{t}^{\mathbf{b_1}}, \ldots, \mathbf{t}^{\mathbf{b_{m'}}}\}$ is a basis of $H_1 \cap H_2$. □

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Computing a basis

Theorem (cont.)

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   1. Compute a basis $\{\mathbf{b_1}, \ldots, \mathbf{b_{m'}}\}$ of $L_1 \cap L_2$,
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   4. compute $S\rho_3^{-1}$, a s.c.r. (right) of $M\rho_3^{-1}$ in F_{n_3},
   5. consider an empty graph with vertices the cosets in $S\rho_3^{-1}$,
   6. $\forall v_k$ generator of $H_1\pi \cap H_2\pi$, and $\forall z_l$ vertex of the graph, search $(M\rho_3^{-1})\, z_l v_k$ and add an edge with label v_k from $(M\rho_3^{-1})\, z_l$ to $(M\rho_3^{-1})\, z_l v_k$ (we obtain the Schreier graph for $M\rho_3^{-1}$ f. i. F_{n_3}),
   7. compute a basis of $M\rho_3^{-1}$ from a maximal tree,
   8. compute a basis $\{\mathbf{t}^{\mathbf{a_1}} v_1, \ldots, \mathbf{t}^{\mathbf{a_{n'}}} v_{n'}\}$ of $(H_1 \cap H_2)\pi\alpha$.
   9. $\{\mathbf{t}^{\mathbf{a_1}} v_1, \ldots, \mathbf{t}^{\mathbf{a_{n'}}} v_{n'}, \mathbf{t}^{\mathbf{b_1}}, \ldots, \mathbf{t}^{\mathbf{b_{m'}}}\}$ is a basis of $H_1 \cap H_2$. □

Computing a basis

Theorem (cont.)

2. in afirmative case, **we can compute a basis** of $H_1 \cap H_2$.
   1. Compute a basis $\{\mathbf{b_1}, \ldots, \mathbf{b_{m'}}\}$ of $L_1 \cap L_2$,
   2. if $(H_1 \cap H_2)\pi = 1$, $\{\mathbf{t}^{\mathbf{b_1}}, \ldots, \mathbf{t}^{\mathbf{b_{m'}}}\}$ is a basis of $H_1 \cap H_2$ (if not, continue).
   3. compute a s.c.r. S of M in $\mathbb{Z}^{n_3}$,
   4. compute $S\rho_3^{-1}$, a s.c.r. (right) of $M\rho_3^{-1}$ in F_{n_3},
   5. consider an empty graph with vertices the cosets in $S\rho_3^{-1}$,
   6. $\forall v_k$ generator of $H_1\pi \cap H_2\pi$, and $\forall z_l$ vertex of the graph, search $(M\rho_3^{-1})\, z_l v_k$ and add an edge with label v_k from $(M\rho_3^{-1})\, z_l$ to $(M\rho_3^{-1})\, z_l v_k$ (we obtain the Schreier graph for $M\rho_3^{-1}$ f. i. F_{n_3}),
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   9. $\{\mathbf{t}^{\mathbf{a_1}} v_1, \ldots, \mathbf{t}^{\mathbf{a_{n'}}} v_{n'}, \mathbf{t}^{\mathbf{b_1}}, \ldots, \mathbf{t}^{\mathbf{b_{m'}}}\}$ is a basis of $H_1 \cap H_2$. □

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   1. Compute a basis $\{\mathbf{b_1}, \ldots, \mathbf{b_{m'}}\}$ of $L_1 \cap L_2$,
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Computing a basis

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Intersection example

Decision of the finite type

Example in $G = F_2 \times \mathbb{Z} = \langle a, b \mid \rangle \times \langle t \mid \rangle$

$H_1 = \langle ta^2, bab^{-1}, t^2 \rangle$
$H_2 = \langle t^2 a^3, ba, t^2 \rangle$

Intersection example
Decision of the finite type

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$H_1 = \langle ta^2, bab^{-1}, t^2 \rangle \ \left[H_1\pi = \langle a^2, bab^{-1} \rangle, \ L_1 = 2\mathbb{Z} \ , \ \mathbf{A_1} = \binom{1}{0} \right]$

$H_2 = \langle t^2 a^3, ba, t^2 \rangle \ \left[H_2\pi = \langle a^3, ba \rangle \ , \ L_2 = 2\mathbb{Z} \ , \ \mathbf{A_2} = \binom{2}{0} \right]$

Intersection example

Decision of the finite type

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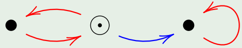
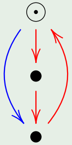
$$H_2 = \langle t^2 a^3, ba, t^2 \rangle \quad \left[H_2\pi = \langle a^3, ba \rangle \;, \; L_2 = 2\mathbb{Z} \;, \; \mathbf{A_2} = \left(\begin{smallmatrix} 2 \\ 0 \end{smallmatrix}\right) \right]$$

$H_1\pi$





$H_2\pi$

Intersection example

Decision of the finite type

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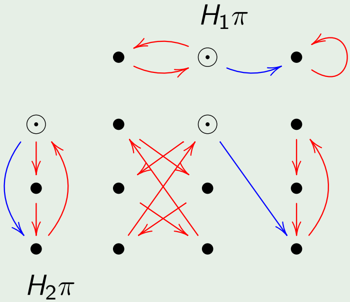
From the *pull-back*,



Intersection example

Decision of the finite type

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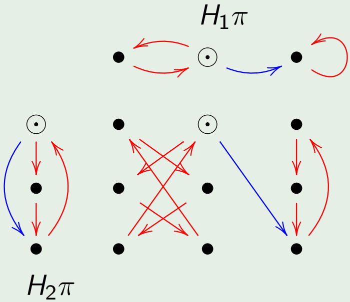
From the *pull-back*, we obtain



Intersection example

Decision of the finite type

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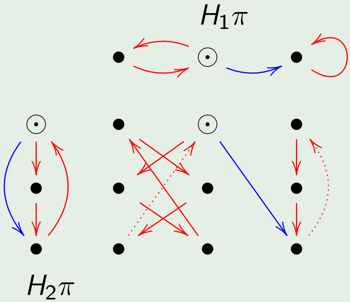
From the *pull-back*, we obtain

- $\{a^6, ba^3b^{-1}\}$ basis of $H_1\pi \cap H_2\pi$,



Intersection example
Decision of the finite type

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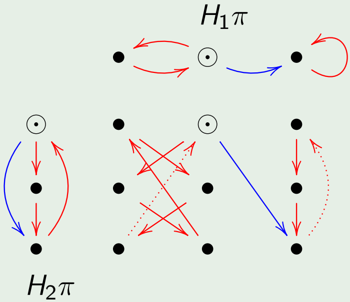
From the *pull-back*, we obtain

- $\{a^6, ba^3b^{-1}\}$ basis of $H_1\pi \cap H_2\pi$,
- $n_3 = 2$,



Intersection example

Decision of the finite type

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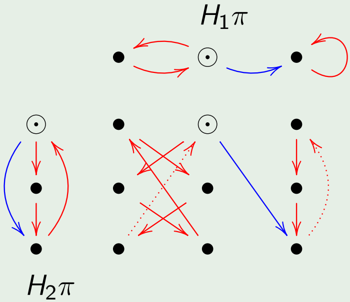

Intersection example

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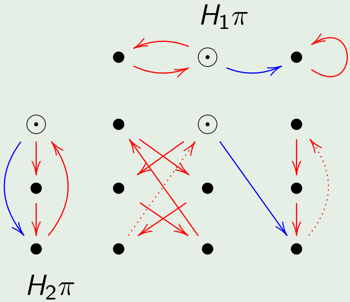

Intersection example

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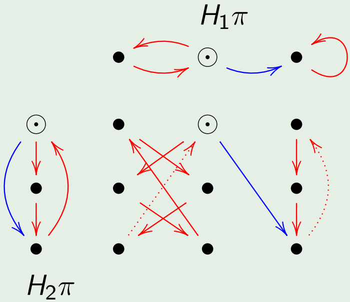

Intersection example

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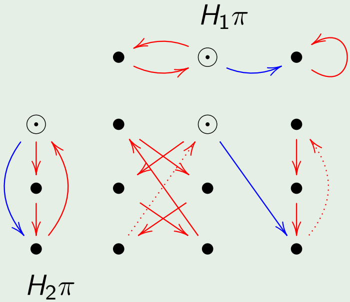

Intersection example
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Then,

- $n_3 \neq 0, 1$ i $M \neq \mathbf{0}$
- $\text{rank}(M) = 2 = n_3$.

Intersection example
Decision of the finite type

Example in $G = F_2 \times \mathbb{Z} = \langle a, b \mid \rangle \times \langle t \mid \rangle$

$$H_1 = \langle ta^2, bab^{-1}, t^2 \rangle \quad \left[H_1\pi = \langle a^2, bab^{-1} \rangle, \; L_1 = 2\mathbb{Z} \;, \; \mathbf{A_1} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right]$$
$$H_2 = \langle t^2a^3, ba, t^2 \rangle \quad \left[H_2\pi = \langle a^3, ba \rangle \;, \; L_2 = 2\mathbb{Z} \;, \; \mathbf{A_2} = \begin{pmatrix} 2 \\ 0 \end{pmatrix} \right]$$

From the *pull-back*, we obtain

- $\{a^6, ba^3b^{-1}\}$ basis of $H_1\pi \cap H_2\pi$,
- $n_3 = 2$,
- $\mathbf{M_1} = \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}, \mathbf{M_2} = \begin{pmatrix} 2 & 0 \\ 1 & 0 \end{pmatrix}$,
- $\mathbf{R_1} = \begin{pmatrix} 3 \\ 0 \end{pmatrix}, \mathbf{R_2} = \begin{pmatrix} 4 \\ 2 \end{pmatrix}$, and
- $M = \langle (2,0), (0,1) \rangle \leqslant_{\text{f.i.}} \mathbb{Z}^2$.

Then,

- $n_3 \neq 0, 1$ i $M \neq \mathbf{0}$
- $\text{rank}(M) = 2 = n_3$.

Therefore,

- $(H_1 \cap H_2)\pi \neq 1$, and

Intersection example

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- $\boldsymbol{H_1 \cap H_2}$ **is f. g.**

Intersection example
Computation of a basis

Example (cont.) $L_1 = L_2 = 2\mathbb{Z}$

- $\{2\}$ is a basis of $L_1 \cap L_2 = 2\mathbb{Z}$,

Intersection example
Computation of a basis

Example (cont.) $M = \langle (2,0),(0,1) \rangle$

- $\{2\}$ is a basis of $L_1 \cap L_2 = 2\mathbb{Z}$,
- $\{(0,0),(1,0)\}$ is a s.c.r. of $\mathbb{Z}^2$ modulus M,

Intersection example

Computation of a basis

Example (cont.) $\rho_3 \colon F_2 \twoheadrightarrow \mathbb{Z}^2$

- $\{2\}$ is a basis of $L_1 \cap L_2 = 2\mathbb{Z}$,
- $\{(0,0),(1,0)\}$ is a s.c.r. of $\mathbb{Z}^2$ modulus M,
- $\{1, v_1\}$ is a s.c.r. (right) of F_2 modulus $M\rho_3^{-1}$,

Intersection example
Computation of a basis

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- construct the Schreier graph for $M\rho_3^{-1}$

Intersection example

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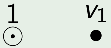

Intersection example
Computation of a basis

Example (cont.)

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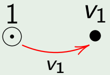

$v_1 \in M\rho_3^{-1} v_1 \Leftrightarrow (0,0) \in M$

Intersection example

Computation of a basis

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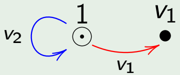

$v_1 \in M\rho_3^{-1} v_1 \Leftrightarrow (0,0) \in M \quad v_2 \in M\rho_3^{-1} \Leftrightarrow (0,1) \in M$

Intersection example

Computation of a basis

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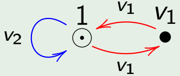

$v_1 \in M\rho_3^{-1} v_1 \Leftrightarrow (0,0) \in M$ $\quad$ $v_2 \in M\rho_3^{-1} \Leftrightarrow (0,1) \in M$

$v_1 v_1 \in M\rho_3^{-1} \Leftrightarrow (2,0) \in M$

Intersection example

Computation of a basis

Example (cont.)

- $\{2\}$ is a basis of $L_1 \cap L_2 = 2\mathbb{Z}$,
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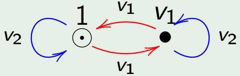

$v_1 \in M\rho_3^{-1} v_1 \Leftrightarrow (0,0) \in M \quad v_2 \in M\rho_3^{-1} \Leftrightarrow (0,1) \in M$

$v_1 v_1 \in M\rho_3^{-1} \Leftrightarrow (2,0) \in M \quad v_1 v_2 v_1^{-1} \in M\rho_3^{-1} \Leftrightarrow (0,1) \in M.$

Intersection example

Computation of a basis

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- $\{2\}$ is a basis of $L_1 \cap L_2 = 2\mathbb{Z}$,
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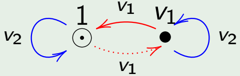

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- we obtain $\{v_2, v_1^2, v_1 v_2 v_1^{-1}\}$ basis of $M\rho_3^{-1} \leqslant F_2$,

Intersection example

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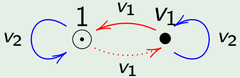

$v_1 \in M\rho_3^{-1} v_1 \Leftrightarrow (0,0) \in M$ $\quad$ $v_2 \in M\rho_3^{-1} \Leftrightarrow (0,1) \in M$

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- $\{ba^3b^{-1}, a^{12}, a^6ba^3b^{-1}a^{-6}\}$ basis of $(H_1 \cap H_2)\pi \leqslant H_1\pi \cap H_2\pi$,

Intersection example

Computation of a basis

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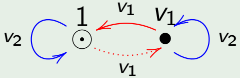

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- $\{ba^3b^{-1}, t^6 a^{12}, a^6 ba^3 b^{-1} a^{-6}\}$ basis of $(H_1 \cap H_2)\pi\alpha \leqslant H_1 \cap H_2$,

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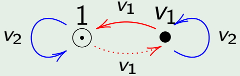

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- $\{ba^3b^{-1}, t^6a^{12}, a^6ba^3b^{-1}a^{-6}, t^2\}$ **basis of $H_1 \cap H_2 \leqslant G$.**

Thanks!

Of course, in the first example, Howson property for F_n says to us that the fact that the intersection of the two free subgroups H and K is not f. g. means that there are not a common free ambient for both.