On the asymptotics of visible elements and homogeneous equations in surface groups

Laura Ciobanu

(University of Fribourg, Switzerland)

with Y. Antolín and N. Viles

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- $ab : F \longrightarrow F/[F, F]$ is the abelianization map.

For any F with $ab(F) = \mathbb{Z}^r$, $g \in F$ is t-visible if $ab(g)$ is t-visible.

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Elements of length $\leq n$ in S: $\rho(n, S) = \sharp\{x \in S : l(x) \leq n\}$.

1. The asymptotic density of S in F is

$$\bar{\rho}(S) = \limsup_{n \to \infty} \frac{\rho(n, S)}{\rho(n, F)}.$$

   If the limit exists, then we denote it by $\rho(S)$ and we call it the *strict asymptotic density*.

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2. Let $S \subseteq F$. The spherical density of S in F is

   $$\bar{\gamma}(S) = \limsup_{n\to\infty} \frac{\gamma(n,S)}{\gamma(n,F)}.$$

   If the limit exists, then we denote it by $\gamma(S)$ and we call it the *strict spherical density.*

3. Let $S \subseteq F$. The annular density of S in F is

   $$\bar{\sigma}(S) = \limsup_{n\to\infty} \frac{1}{2}\left(\frac{\sharp\{x\in S : l(x)=n-1\}}{\sharp\{x\in F : l(x)=n-1\}} + \frac{\sharp\{x\in S : l(x)=n\}}{\sharp\{x\in F : l(x)=n\}}\right)$$

   If the limit exists, then we denote it by $\sigma(S)$ and we call it the *strict annular density.*

Comments

For $1 \leq p \leq \infty$, let $l_p \colon \mathbb{Z}^r \to \mathbb{R}$ denote the restriction to $\mathbb{Z}^r$ of the $||.||_p$-norm from $\mathbb{R}^r$ and write $\rho_p(S)$ or $\rho_\infty(S)$ etc.

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Proposition (Christopher, 1956) Let $t \geq 1$, $t \in \mathbb{Z}$, and let U_t be the set of t-visible elements in $\mathbb{Z}^r$. Then

$$\rho_\infty(U_t) = \frac{1}{t^r \zeta(r)}.$$

Theorem (Kapovich, Rivin, Schupp, Shpilrain, 2007)

Let F be a free group of rank k with generating set A and abelianization $ab : F \longrightarrow \mathbb{Z}^k$. Let $S \subset \mathbb{Z}^k$ be an $SL(k, \mathbb{Z})$-invariant subset and $\tilde{S} = ab^{-1}(S) \subset F$. Then:

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1. For all $1 \leq p \leq \infty$, $\rho_p(S)$ exists and $\boxed{\rho_p(S) = \rho_\infty(S).}$
2. The annular density $\sigma_A(\tilde{S})$ exists, and $\boxed{\sigma_A(\tilde{S}) = \rho_\infty(S).}$
3. Let V_t be the set of t-visible elements in F.

Then $\boxed{\sigma_A(V_t) = \frac{1}{t^k \zeta(k)}}$ and $\boxed{0 < \limsup_{n\to\infty} \frac{\rho_A(n,V_t)}{\rho_A(n,F)} < 1.}$

Key points

For free groups:

- The annular density of t-visible elements exists and is equal to $\frac{1}{t^r\zeta(r)}$.

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- The asymptotic density of visible elements does NOT exist.

- The spherical density does NOT exist, BUT one can look at the visible elements of even and odd lengths separately.

Theorem (Kapovich, Rivin, Schupp, Shpilrain, 2007)

Let $k = 2$, i.e. F is the free group of rank 2. Then

$$\lim_{m\to\infty} \frac{\gamma_A(2m, V_1)}{\gamma_A(2m, F)} = \frac{2}{3\zeta(2)} = \frac{4}{\pi^2}$$

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$$\lim_{m\to\infty} \frac{\gamma_A(2m-1, V_1)}{\gamma_A(2m-1, F)} = \frac{8}{\pi^2}.$$

Theorem 1 (Antolin, C, Viles, 2010)

Let $k \geq 2$ and G be the free group of rank k or the surface group of genus k. Let r be the rank of the abelianization of G. ($r = k$ or $r = 2k$)

Then

$$\lim_{m\to\infty} \frac{\gamma_A(2m, V_1)}{\gamma_A(2m, G)} = \frac{2^r - 2}{(2^r - 1)\zeta(r)}$$

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Free group of rank k:

$$F_k = \langle a_1, \ldots, a_k \mid \rangle$$

Surface group of genus k:

$$S_k = \langle a_1, \ldots, a_k \mid [a_1, b_1] \ldots [a_k, b_k] \rangle$$

The proof of Theorem 1 relies on:

- free and surface groups satisfy Hypothesis $\circledast$

and

- Theorem 2.

Hypothesis $\circledast$

Let G be a group generated by a finite set A with abelianization $\mathbb{Z}^r$ and $ab : G \to \mathbb{Z}^r$ the abelianization map. G satisfies $\circledast$ if

there exists a symmetric positive definite real matrix D such that

$$\lim_{n\to\infty} \left| (\det D)^{1/2} n^{r/2} \left(\frac{\gamma_A(n, ab^{-1}(\alpha))}{\gamma_A(n, G)} + \frac{\gamma_A(n+1, ab^{-1}(\alpha))}{\gamma_A(n+1, G)} \right) - \frac{2}{(2\pi)^{r/2}} e^{-\langle \alpha, D^{-1}\alpha\rangle/2n} \right| = 0,$$

uniformly in $\alpha \in \mathbb{Z}^r$.

R. Sharp, *Local limit theorems for free groups* (2001)

Let G be the free group of rank k or surface group of genus k, and A and r be the corresponding generating set and rank of the abelianization of G. Let $ab : G \to \mathbb{Z}^r$ be the abelianization map.

Then G satisfies **Hypothesis** $\circledast$.

Theorem 2 (Antolin, C, Viles, 2010)

Let G, A, r satisfy Hypothesis $\circledast$, $S \subseteq \mathbb{Z}^r$ be a H_r-invariant subset and $\tilde{S} = ab^{-1}(S)$, where

$$H_r = \{M \in SL(r, \mathbb{Z}) \mid M = I_r \text{ in } SL(r, \mathbb{Z}/2\mathbb{Z})\}.$$

1. The strict annular density $\sigma_A(\tilde{S})$ exists and, $\boxed{\sigma_A(\tilde{S}) = \rho_\infty(S).}$

2. Let $U_1 =$ set of visible elements in $\mathbb{Z}^r$ and $V_1 = ab^{-1}(U_1) =$ the visible elements in G. Let $U_1^{ev} = \{z \in U_1 : l_1(z) \text{ is even}\}$ denote the visible elements of even length.

If $ab^{-1}(U_1^{ev}) = \{v \in V_1 : |v|_A \text{ is even}\}$, then

$$\lim_{m\to\infty} \frac{\gamma_A(2m, V_1)}{\gamma_A(2m, G)} = 2\rho_\infty(U_1^{ev}) = \frac{2^r - 2}{(2^r - 1)\zeta(r)},$$

$$\lim_{m\to\infty} \frac{\gamma_A(2m-1, V_1)}{\gamma_A(2m-1, G)} = 2\rho_\infty(U_1) - 2\rho_\infty(U_1^{ev}) = \frac{2^r}{(2^r - 1)\zeta(r)}.$$

Proof of Theorem 2

(i) Technical ... transform Hypothesis $\circledast$ into a statement about probability distributions and their covergence.

(ii) U_1 is H_r-invariant $\Rightarrow U_1^{ev}$ is H_r-invariant.

Take $S = U_1^{ev}$, n even. Let

$$Q(n) = \frac{\gamma(n-1, ab^{-1}(S))}{2\gamma(n-1, G)} + \frac{\gamma(n, ab^{-1}(S))}{2\gamma(n, G)}$$

$$= \frac{\gamma(n-1, ab^{-1}(S))}{2\gamma(n-1, G)}.$$

By (i), $2\lim_{m\to\infty} Q(2m) = \frac{\gamma(2m,V_1)}{\gamma(2m,G)} = 2\rho_\infty(U_1^{ev})$ and

Proposition (ACV 2010):

$$\rho_\infty(U_1^{ev}) = \frac{2^{r-1}-1}{2^r-1}\rho_\infty(U_1) = \frac{2^{r-1}-1}{(2^r-1)\zeta(r)}.$$

Remember that U_1 is the set of visible elements in $\mathbb{Z}^r$.

qed

Motivation:

Understand the asymptotic behavior of homogeneous equations in free and surface groups.

Equations in groups

- G - a group (infinite and non-commutative)
- $X = \{X_1, \ldots, X_n\}$ - a set of variables.

An *equation* in variables $X_1, \ldots, X_n$ with coefficients g_j in G is a formal expression of the form

$$g_1 X_{i_1}^{\epsilon_1} g_2 X_{i_2}^{\epsilon_2} \ldots X_{i_m}^{\epsilon_m} g_{m+1} = 1$$

where $\epsilon_j \in \{1, -1\}$.

Homogeneous equations

Equations in which all the variables appear on one side and all the coefficients on the other side:

$$X_{i_1}^{\epsilon_1} X_{i_2}^{\epsilon_2} \ldots X_{i_m}^{\epsilon_m} = w.$$

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Let $z(X_1, \ldots, X_n) = X_{i_1}^{\epsilon_1} X_{i_2}^{\epsilon_2} \ldots X_{i_m}^{\epsilon_m}$. The above equation has solutions iff there exists an homomorphism $\phi \colon F_n \to G$ such that $\phi(z(x_1, \ldots, x_n)) = w$.

- Example: $W = [a, b]$, $U = a^2$. Does there exist ϕ such that
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Let $\phi(a) = X$ and $\phi(b) = Y$. Then $\phi(W) = U$ if and only if the equation has solutions.

$$XYX^{-1}Y^{-1} = a^2$$

Since there are no solutions, no endomorphism sends W to U.

Question: what happens when m and $|w|_A$ go to infinity?

$$X_{i_1}^{\epsilon_1} X_{i_2}^{\epsilon_2} \dots X_{i_m}^{\epsilon_m} = w.$$

The spherical (s,t)-mapping ratio $e_\gamma(F,G,s,t)$:

$$\boxed{l_F(f) = s,\ l_G(g) = t}$$

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$$e_\gamma(F,G,s,t) = \frac{\sharp\{(f,g) \in F \times G : l_F(f) = s, l_G(g) = t, \phi(f) = g \text{ for some } \phi \in \mathrm{Hom}(F,G)\}}{\sharp\{(f,g) \in F \times G : l_F(f) = s, l_G(g) = t\}}$$

Theorem (A, C, V, 2010)

Let G_k and G_n be free or surface groups and let A, B be their respective generating sets. Let $r(k)$ and $r(n)$ denote the ranks of the abelianization of G_k and G_n, respectively. Let $\epsilon, \delta \in \{0, 1\}$. Then:

$$\frac{2^{r(n)} - 2(1-\epsilon)}{(2^{r(n)}-1)\zeta(r(n))} \leq \liminf_{s\to\infty, t\to\infty} e_\gamma(G_n, G_k, 2s+\epsilon, 2t+\delta),$$

$$\limsup_{s\to\infty, t\to\infty} e_\gamma(G_n, G_k, 2s+\epsilon, 2t+\delta) \leq 1 - \frac{2^{r(k)} - 2(1-\delta)}{(2^{r(k)}-1)\zeta(r(k))}\left(1 - \frac{2^{r(n)} - 2(1-\epsilon)}{(2^{r(n)}-1)\zeta(r(n))}\right).$$

Corollary. Let G be a surface group of genus ≥ 2 or a free group of rank ≥ 2.

Let

$$A(s,t) = \frac{\sharp\{\text{solvable } (s,t)\text{-homogeneous equations in } G \text{ in } n \text{ variables}\}}{\sharp\{(s,t)\text{-homogeneous equations in } G \text{ in } n \text{ variables}\}}.$$

Then

$$0 < \liminf_{s\to\infty, t\to\infty} A(s,t) \leq \limsup_{s\to\infty, t\to\infty} A(s,t) < 1.$$

Connection between visible elements and homogeneous equations

Lemma. Let F, G be groups whose abelianization is free-abelian of finite rank. Let $f \in F$.

1. Let $\phi : F \to G$ be a group homomorphism. Then $\gcd(ab(\phi(f)))$ is a multiple of $\gcd(ab(f))$. In particular, if $\gcd(ab(f)) = \infty$, then $\gcd(ab(\phi(f))) = \infty$.

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1. Let $\phi : F \to G$ be a group homomorphism. Then $\gcd(ab(\phi(f)))$ is a multiple of $\gcd(ab(f))$. In particular, if $\gcd(ab(f)) = \infty$, then $\gcd(ab(\phi(f))) = \infty$.

2. If, moreover $\gcd(ab(f)) = 1$, then for any element g in G there exists a homomorphism $\phi\colon F \to G$ such that $\phi(f) = g$.

In other words:

1. Homomorphisms between groups with free-abelian abelianization (of finite rank) send t-visible elements to (tm)-visible elements, where t, m are positive integers.

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1. Homomorphisms between groups with free-abelian abelianization (of finite rank) send t-visible elements to (tm)-visible elements, where t, m are positive integers.

2. A visible element can be mapped to any element via a homomorphism!!!

Work of Gilman, Myasnikov, Roman'kov:

F = free group of rank m

- $V(F,k)$ = the set of all homogeneous equations in k var's in F.

- $V^+(F,k)$ = the set of all satisfiable homogeneous equations in k var's in F.

Theorem (2010). For $c(k) = \frac{2k-3}{(k-1)(2k-1)}$:

$$c(k)\frac{1}{\zeta(k)} \geq \liminf_{r\to\infty} \rho_r(V^+|V),$$

$$\limsup_{r\to\infty} \rho_r(V^+|V) \leq 1 - \frac{c(k)c(m)}{2k-1}\frac{\zeta(k)-1}{\zeta(k)\zeta(m)}.$$

Theorem (GMR 2010). The set of satisfiable one-variable equations in free groups is negligible relative to the spherical, as well as the ball, stratification.

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For finitely generated nilpotent groups with ball stratification:

Theorem (GMR 2010). The set of satisfiable equations is intermediate.

Theorem (GMR 2010). The set of satisfiable one-variable equations is negligible relative.

Generalizations

1. More groups?

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2. More equations?

Thank you!