Bredon homology of groups with cyclic torsion

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Outline

Homology theories

An homology theory is just a sequence of functors

$$H_i \colon \mathfrak{Top} \to \mathfrak{Ab} \quad i \in \mathbb{Z}$$

+ some axioms

An homology theory for groups is "usually" an homology theory applied to some space associated to a group

$$\mathfrak{Groups} \to \mathfrak{Top} \xrightarrow{H_i} \mathfrak{Ab}.$$

Naive example

The singular homology of a simplicial complex X

$H_n(X) = \mathbb{Z}^i$ where i is the number of n-th dimensional holes

Non-naive examples

$K_i^G(\underline{E}\,G)$	G-equivariant K-homology of the classifying space for proper actions.
$K_i^{top}(C_r^*G)$	Analytic K-theory of the reduced C^*-algebra of G.
$H_i^{\mathfrak{Fin}}(\underline{E}\,G; R_{\mathbb{C}})$	Bredon homology groups for the classifying space for proper actions with respect to the complex representation ring

The interest of Bredon homology groups

The Bredon homology groups $H_i^{\mathfrak{Fin}}(\underline{E}\,G; R_{\mathbb{C}})$ can be used to compute (via spectral sequences) the G-equivariant homology groups $K_i^G(\underline{E}\,G)$. These groups appear in the context of the

Baum-Connes conjecture

Let G be a second countable locally compact group. The assembly map

$$\mu_G \colon K_i^G(\underline{E}\,G) \to K_i^{top}(C_r^* G) \quad i = 1, 2$$

is an isomorphism.

Usually, the left part is "easier" to compute than the right part.

Summary

- In [Mislin] Guido Mislin computed the Bredon homology of one-relator groups with coefficients in the complex representation ring and used it to describe the left part of the Baum-Connes Conjecture.
- In [AF], we distilled the proof of Mislin and apply it to a family of groups $\mathcal{G}_{\text{cct}}$ with controlled cyclic torsion.
- We show many examples of groups in $\mathcal{G}_{\text{cct}}$, which include Hempel groups (a generalized version surface-plus-one-relator groups).
- From existing results, we conclude that Hempel groups satisfy the Baum-Connes conjecture, and we describe the left part.

[AF] Yago Antolin and Ramón Flores *On groups with cyclic torsion* arXiv:1107.0566 Preprint 2011.

[Mislin] Guido Mislin, *Equivariant K-homology of the classifying space for proper actions*, Adv. Courses Math. CRM Barcelona, Birkhäuser, Basel, 2003.

Classifying space for free actions

Let G be a discrete group.

A *model for the classifying space for free G-actions EG* is a contractible G-CW-complex X where G acts freely.

Recall that X is *contractible* if it is simply connected and acyclic.

Two models for the classifying space are homotopic.

The quotient $G\backslash X$ is a model for BG (or $K(G,1)$) and has fundamental group G.

The cellular structure of X and the free G-action gives a resolution of $\mathbb{Z}$ by free $\mathbb{Z}G$-modules

$$\cdots \to \bigoplus_{\text{2-cells } G\backslash X} \mathbb{Z}G \to \bigoplus_{\text{1-cells of } G\backslash X} \mathbb{Z}G \to \bigoplus_{\text{0-cells of } G\backslash X} \mathbb{Z}G \to \mathbb{Z} \to 0$$

Cayley complex

The *Cayley complex* of a presentation $\langle\, X \parallel R \,\rangle$ is a 2-dimensional G-CW-complex where the 1-skeleton is the Cayley graph of $\langle\, X \parallel R \,\rangle$ and there is one G-orbit of 2-cells for every $r \in R$; the attachment is following the word r on the Cayley graph starting at the vertex g.

The Cayley complex is always simply connected!

The Cayley complex $\langle\, X \parallel R \,\rangle$ is a model for $E(\langle\, X \mid R \,\rangle)$ if and only if is it is acyclic.

Example

The Cayley complex of $\langle\, a, b \parallel [a, b] \,\rangle$ is a model for $E(\mathbb{Z} \times \mathbb{Z})$.

$\mathbb{Z}/n\mathbb{Z}$

The classifying space for a finite group is infinite dimensional!!!

However, by Group Theory, $\mathbb{Z}/n\mathbb{Z}$ should be thought as the rotations of a regular n-gon (*a two dimensional object*).

The action is "almost" free. There is only one point (the center of the n-gon) where $\mathbb{Z}/n\mathbb{Z}$ does not act trivially.

Classifying space for proper actions

A *model for the classifying space for proper G-actions* $\underline{E}\,G$ is a contractible G-CW-complex X such that for all $H \leqslant G$

- If H is infinite, H fixes no point of X.
- If H is finite, the subcomplex X^H of fixed points is contractible.

Two models for the classifying space are homotopic.

The quotient $G\backslash X$ is a model of $\underline{B}G$. It is an orbifold whose fundamental group is G.

The cellular structure of X and the G-action gives a resolution of $\mathbb{Z}$ by $\mathbb{Z}G$-modules

$$\cdots \to \bigoplus_{x \text{ a 1-cells of } G\backslash X} \mathbb{Z}[G/G_x] \to \bigoplus_{x \text{ a 0-cells of } G\backslash X} \mathbb{Z}[G/G_x] \to \mathbb{Z} \to 0$$

Here G_x denotes the G-stabilizer of x.

Dimension of $\underline{\mathrm{E}}\, G$.

- $\dim \underline{\mathrm{E}}\, G$ is the minimum of the dimensions of the models of $\underline{\mathrm{E}}\, G$.
- If X is a model for $\underline{\mathrm{E}}\, G$, X^{sing} is the subcomplex with non-trivial stabilizers.
- $\dim \underline{\mathrm{E}}\, G^{\text{sing}}$ is the minimum of the dimensions of the models of $\underline{\mathrm{E}}\, G^{\text{sing}}$.

Example: G a finite group

- $\dim EG = \infty$.
- $\dim \underline{\mathrm{E}}\, G = 0$.

Example: G finitely generable and $\dim \underline{\mathrm{E}}\, G \leq 1$

In this case, there is a model that is a tree.
Then G acts on a tree with finite stabilizers and hence G is virtually free.
Conversely, by a theorem of Dunwoody (1979), if G is virtually free then it acts on a tree with finite stabilizers, and hence $\dim \underline{\mathrm{E}}\, G \leq 1$.

Condition (C)

We are interested in the groups such that both $(\underline{\mathrm{E}}\,G)^{sing}$ and the torsion are as simple as possible.

Precisely we deal with groups satisfying the following condition:

(C) There is a finite family of non-trivial finite cyclic subgroups $\{G_\lambda\}_{\lambda\in\Lambda}$such that for each non-trivial torsion subgroup H of G, there exists a unique $\lambda \in \Lambda$ and a unique coset $gG_\lambda \in G/G_\lambda$ such that $H \leqslant gG_\lambda g^{-1}$.

In particular, the groups G_λ are maximal *malnormal* in G.

The class of groups G satisfying the condition (C) will be denoted by $\mathcal{G}_{cct}$.

The easy examples

- Any finite cyclic group and *any torsion-free group* is in $\mathcal{G}_{cct}$.
- The class is also *closed by free products*, so for example, the infinite dihedral group is in $\mathcal{G}_{cct}$.

The dimension of $\underline{\mathrm{E}}\, G^{\mathrm{sing}} = 0$

Proposition

Let G and $\{G_\lambda\}_{\lambda\in\Lambda}$ satisfying (C). Then $\dim(\underline{\mathrm{E}}\, G^{sing}) = 0$.

Proof.

Let $X = G \sqcup \{gG_\lambda : \lambda \in \Lambda, g \in G\}$, a left G-action.
We use the bar construction to build a simplicial complex $\mathcal{C}(X)$.
The n-simplices of $\mathcal{C}(X)$ are the $(n+1)$-tuples $(x_0, \ldots, x_n)$ of elements of X with $x_i \neq x_j$ if $i \neq j$.
An n-simplex $(x_0, \ldots, x_n)$ is attached to the $(n-1)$-simplices $(x_0, \ldots, \widehat{x_i}, \ldots, x_n)$.
The space $\mathcal{C}(X)$ is contractible.
G acts freely on $G \subseteq X$, and, a subgroup H of G fixes gG_λ if and only $g^{-1}Hg \subseteq G_\lambda$ and hence H is finite. By condition (C), H fixes exactly one element of X, and therefore, H fixes only one vertex of $\mathcal{C}(X)$.
Then $\mathcal{C}(X)$ is an $\underline{\mathrm{E}}\, G$, and $\dim(\mathcal{C}(X)^{sing}) = 0$. ☐

Complex representation ring

A *complex representation of G* is a group homomorphism

$$\rho\colon G \to GL(V),$$

where V is a finite dimensional, $\mathbb{C}$-linear space, and $GL(V)$ is the group of linear isomorphisms of V.

The underlying abelian group

Given $\rho_1 : G \to GL(V_1)$ and $\rho_1 : G \to GL(V_2)$, the direct sum is

$$\rho_1 \oplus \rho_2 : G \to GL(V_1 \oplus V_2), \qquad (\rho_1 \oplus \rho_2)(g)(v, w) = (\rho_1(g)(v), \rho_2(g)(w)).$$

The isomorphism classes of fin. dim. linear $\mathbb{C}$-representations with $\oplus$ form an *abelian semigroup*. It can be *extended to a group* by considering formal differences of isomorphism classes.

This the underlying abelian group of *complex representation ring*, denoted $R_{\mathbb{C}}(G)$. (The multiplication is given by direct product of representations).

Representations ring for finite cyclic groups

Lemma

Let C_n denote the cyclic group of order n. Then $R_{\mathbb{C}}(C_n)$ is isomorphic to $\mathbb{Z}[x]/(x^n - 1)$, where x corresponds to the complex representation sending a generator of the group to a primitive n-th root of unity. ☐

0-th Bredon homology group

The 0-th Bredon homology group with coefficients in $R_{\mathbb{C}}$ is given by:

$$\mathrm{H}_0^{\mathfrak{F}}(G; R_{\mathbb{C}}) = \operatorname{colim}_{K \leqslant G, \text{ finite}} R_{\mathbb{C}}(K).$$

The main result

Theorem (Mislin, A-Flores)

Let G be a group on the class $\mathcal{G}_{cct}$ and let $\{G_\lambda\}_{\lambda\in\Lambda}$ be the subgroups for which condition (C) holds. Then

(i). $\mathrm{H}_i^{\mathfrak{F}}(G; R_{\mathbb{C}}) = \mathrm{H}_i(\underline{B}G; \mathbb{Z})$ *for* $i \geq 2$.

(ii). $\mathrm{H}_1^{\mathfrak{F}}(G; R_{\mathbb{C}}) = (G/\operatorname{Tor}(G))_{ab}$ *where* $\operatorname{Tor}(G)$ *denotes the subgroup of G generated by the torsion elements.*

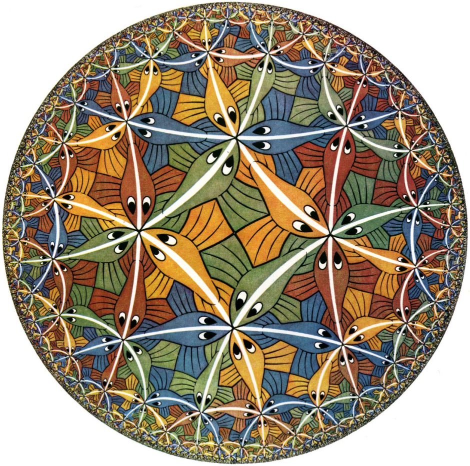
(iii). $\mathrm{H}_0^{\mathfrak{F}}(G; R_{\mathbb{C}}) = \begin{cases} \prod_{\lambda\in\Lambda} R_{\mathbb{C}}(G_\lambda), & \text{if } \Lambda \neq \emptyset \\ \mathbb{Z} & \text{if } \Lambda = \emptyset. \end{cases}$

Extensions

Extensions $1 \to \mathbb{Z}^n \to G \to C \to 1$, where C is a finite cyclic group and the conjugation action of C in $\mathbb{Z}^n$ is free outside $0 \in \mathbb{Z}^n$ lie in $\mathcal{G}_{cct}$.

NEC-groups without finite dihedral subgroups

A non-Euclidean crystallographic group (NEC group) is a discrete group of isometries of the hyperbolic plane.



One-relator quotients of locally indicable groups

Let A, B be *locally indicable groups*. Let $r \in A * B$, such that r is not conjugate to an element of A nor of B. Let $G = (A * B)/\langle\!\langle r \rangle\!\rangle$. Let $G_r = \langle \sqrt[A*B]{r} \rangle / \langle r \rangle \leq G$.

Theorem (Howie)

- *A model of BG can be constructed from models of BA, BB and BG_r.*
- *G and G_r satisfy condition (C).*

We use Howie's construction to describe a model of $\underline{E}\,G$.

Aspherical groups

Let $F := \langle\, X \mid \ \rangle$ and $G := \langle\, X \mid R \,\rangle$.Let $G_r := \langle\, \sqrt[F]{r} \,\rangle / \langle\, r \,\rangle \leq G$, a finite cyclic subgroup of G of order $\log_F(r)$.

There exist several concepts of aspherical presentations. We will say that $\langle\, X \parallel R \,\rangle$ is *an aspherical presentation* if the abelianized of $\langle\, {}^F R \,\rangle$ is isomorphic to $\oplus_{r \in R} \mathbb{Z}[G/G_r]$. A group is aspherical if it admits an aspherical presentation.

Theorem (Lyndon)

One-relator presentations are aspherical

$\underline{E}\, G$ of aspherical presentations

An $\underline{E}\, G$ can be constructed from the Cayley complex of an aspherical presentation $\langle\, X \parallel R \,\rangle$, by changing the orbit of 2-cells corresponding to r by a G/G_r-orbit of $\underline{E}\, G_r$.

▸ More aspherical

Hempel groups

Hempel and Howie studied (orientable) *surface-plus-one relation groups*, i.e. groups with presentation

$$\langle\, x_1, y_1, \ldots, x_g, y_g \parallel [x_1, y_x][x_2, y_2] \cdots [x_n, y_n], r \,\rangle.$$

By a *Hempel group* we mean a group with presentation

$$\langle\, x, y, z_1, \ldots, z_n \parallel [x, y]s, r \,\rangle$$

where $s \in \langle\, z_1, \ldots, z_n \,\rangle$ and r “is not a power of x.”

Theorem (A-Dicks-Linnell)

1. *A Hempel presentation is aspherical.*
2. *A Hempel groups is an HNN-extension of a one-relator group and the associated subgroups are Magnus subgroups.*

Corollary (A-Flores)

Hempel groups are Cohen-Lyndon aspherical.

Hempel groups and the BCC conjecture

The conjecture BCC has been verified for the class $\mathbf{LH}\mathcal{TH}$, which contains soluble groups, finite groups and it is closed under passing to subgroups and HNN-extensions.

The previous theorem implies that Hempel groups are in $\mathbf{LH}\mathcal{TH}$.

Proposition (A-Flores)

Let G be given by a Hempel presentation

$$\langle x_1, \ldots, x_k \parallel w, r \rangle.$$

Let $F = \langle x_1, \ldots, x_k \mid \quad \rangle$ and $G_r := \langle \sqrt[F]{r} \mid r \rangle$. Then

1. $K_i^G(\underline{E}\, G) \simeq K_i^{top}(C_r^*(G))$ for $i = 0, 1$.
2. There is a split short exact sequence
$$R_{\mathbb{C}}(G_r) \to K_0^G(\underline{E}\, G) \to (\langle {}^F r \cup {}^F w \rangle \cap [F, F])/[F, \langle {}^F r \cup {}^F w \rangle]$$
3. There is a natural isomorphism $(\langle x_1, \ldots, x_k \mid w, \sqrt[F]{r} \rangle)_{\text{ab}} \simeq K_1^G(\underline{E}\, G)$.

The orbit category

Let G be a discrete group and $\mathfrak{F}$ the family of finite subgroups of G.

The *orbit category* $\mathfrak{D}_{\mathfrak{F}}(G)$ is the category whose objects are left coset spaces G/K with $K \in \mathfrak{F}$, and morphism sets $\mathrm{mor}(G/K, G/L)$ given by the G-maps $G/K \to G/L$.

G-Mod$_{\mathfrak{F}}$

Let G-Mod$_{\mathfrak{F}}$ and Mod$_{\mathfrak{F}}$-G be respectively the category of covariant and contravariant functors

$$\mathfrak{O}_{\mathfrak{F}}(G) \to \mathfrak{Ab}.$$

Morphisms in G-Mod$_{\mathfrak{F}}$ and Mod$_{\mathfrak{F}}$-G are natural transformations of functors.

Example: G torsion-free

Notice that if $\mathfrak{F}$ consists only on the trivial group then G-Mod$_{\mathfrak{F}}$ and Mod$_{\mathfrak{F}}$-G are the usual categories of right and left $\mathbb{Z}G$-modules.

Homology

Now $\mathrm{Tor}_i(-, N)$ is defined as the i-th left derived functor of the *categorical tensor product functor*

$$- \otimes_{\mathfrak{F}} N \colon \ \mathrm{Mod}_{\mathfrak{F}}\text{-}G \to \mathfrak{Ab}$$

The *Bredon homology groups of G with coefficients in N* $\in G\text{-}\mathrm{Mod}_{\mathfrak{F}}$ are given by

$$\mathrm{H}_i^{\mathfrak{F}}(G; N) := \mathrm{Tor}_i(\underline{\mathbb{Z}}, N), \ i \geq 0,$$

where $\underline{\mathbb{Z}}$ denotes the constant functor which assigns to each object the abelian group $\mathbb{Z}$.

Given and orbit space G/K, we are interested in the functor that assigns to G/K the complex representation ring of K.

The main result revisited

Theorem (Mislin, A-Flores)

Let G be a group in the class $\mathcal{G}_{cct}$ and let $\{G_\lambda\}_{\lambda\in\Lambda}$ be the subgroups for which condition (C) holds. Then

(i). $\mathrm{H}_i^{\mathfrak{F}}(G; R_{\mathbb{C}}) = \mathrm{H}_i(\underline{B}G; \mathbb{Z})$ *for* $i \geq 2$.

(ii). $\mathrm{H}_1^{\mathfrak{F}}(G; R_{\mathbb{C}}) = (G/\operatorname{Tor}(G))_{ab}$ *where* $\operatorname{Tor}(G)$ *denotes the subgroup of G generated by the torsion elements.*

(iii). $$\mathrm{H}_0^{\mathfrak{F}}(G; R_{\mathbb{C}}) = \begin{cases} \prod_{\lambda\in\Lambda} R_{\mathbb{C}}(G_\lambda), & \text{if } \Lambda \neq \emptyset \\ \mathbb{Z} & \text{if } \Lambda = \emptyset. \end{cases}$$

Corollary (Bredon Homology for aspherical groups)

Let G be an aspherical groups with aspherical presentation $\langle\, X \parallel R\,\rangle$, For each $r \in R$, let G_r be the cyclic subgroup of G generated by the image of $\sqrt[F]{r}$.
Then

(i). $\mathrm{H}_i^{\mathfrak{F}}(G; R_{\mathbb{C}}) = 0$ *for* $i > 2$.

(ii). $\mathrm{H}_2^{\mathfrak{F}}(G; R_{\mathbb{C}}) = \mathrm{H}_2(G; \mathbb{Z})$.

(iii). $\mathrm{H}_1^{\mathfrak{F}}(G; R_{\mathbb{C}}) = (G/\operatorname{Tor}(G))_{ab}$ *where* $\operatorname{Tor}(G)$ *denotes the subgroup of G generated by the torsion elements.*

(iv). $$\mathrm{H}_0^{\mathfrak{F}}(G; R_{\mathbb{C}}) = \begin{cases} \prod_{r\in R, G_r \neq \{1\}} R_{\mathbb{C}}(G_r), & \text{if } \exists G_r \neq \{1\}\emptyset \\ \mathbb{Z} & \text{if } G_r = \{1\} \forall r \in R. \end{cases}$$

Why is our result SUPER-MEGA-GREAT?

By results of Arzhantseva and Ol'shanskii, generically finitely presented groups are *torsion-free aspherical.*

Generically, our result apply to all f.p. groups!

In particular, the left part of the Baum-Connes conjecture is generically easy to compute.

Thank you!