Dicks' simplification of Mineyev's proof of the SHNC
(or how to remove a 2)

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Outline

Notation

- F will denote the free group on $\{a, b\}$.
- G, H, K will denote subgroups of F.
- Let Y be a graph
  - VY and EY are the vertex and edges of Y.
  - Let Y_0 be any maximal subtree of Y.
  - $\mathrm{r}(Y) = |EY| - |EY_0|$.
  - $\chi(Y) = |VY| - |EY| = 1 - \mathrm{r}(Y)$.
  - $\bar{\mathrm{r}}(Y) = \max(0, \mathrm{r}(Y) - 1) = \max(0, -\chi(Y))$.
- r, χ and $\bar{\mathrm{r}}$ are invariant under homotopy equivalence.
- If $G = \pi_1(Y)$, we define $\mathrm{r}(G) = \mathrm{r}(Y)$, $\chi(G) = \chi(Y)$ and $\bar{\mathrm{r}}(G) = \bar{\mathrm{r}}(Y)$.

Hanna Neumann (1914-1971)

You all know her!

HNN

Howson's and Hanna Neumann results

- Howson (1954) proved

$$\mathrm{r}(H \cap K) \leq \mathrm{r}(H)\,\mathrm{r}(K) + \bar{\mathrm{r}}(H)\,\bar{\mathrm{r}}(K).$$

- Howson also observed that, taking
  - $H = \langle\langle a, b^n \rangle\rangle$, then $\bar{\mathrm{r}}(H) = n$.
  - $K = \langle\langle a^m, b \rangle\rangle$, then $\bar{\mathrm{r}}(K) = m$.

  then

$$\bar{\mathrm{r}}(H \cap K) = \bar{\mathrm{r}}(H)\,\bar{\mathrm{r}}(K).$$

- Hanna Neumann (1956) improved

$$\bar{\mathrm{r}}(H \cap K) \leq 2\,\bar{\mathrm{r}}(H)\,\bar{\mathrm{r}}(K).$$

The Hanna Neumann Conjecture

In 1956 Hanna Neumann asked

The Hanna Neumann Conjecture (HNC)

$$\bar{\mathrm{r}}(H \cap K) \leq \bar{\mathrm{r}}(H)\,\bar{\mathrm{r}}(K).$$

In 1990 Walter Neumann formulated

The Strengthened Hanna Neumann Conjecture (SHNC)

$$\sum_{x \in H \backslash F / K} \bar{\mathrm{r}}(x^{-1}Hx \cap K) \leq \bar{\mathrm{r}}(H)\,\bar{\mathrm{r}}(K).$$

Observe that in the previous example the bounds are realised.

Results

- Burns (1971)

$$\bar{\mathrm{r}}(H \cap K) \leq 2\,\bar{\mathrm{r}}(H)\,\bar{\mathrm{r}}(K) - \min\{\bar{\mathrm{r}}(H), \bar{\mathrm{r}}(K)\}.$$

- W. Neumann (1990)

$$\sum_{x \in H \backslash F / K} \bar{\mathrm{r}}(x^{-1}Hx \cap K) \leq 2\,\bar{\mathrm{r}}(H)\,\bar{\mathrm{r}}(K) - \min\{\bar{\mathrm{r}}(H), \bar{\mathrm{r}}(K)\}.$$

- Tardos (1992) SNHC holds for $\mathrm{r}(H) \leq 2$.
- Dicks (1994) Equivalence between the SHNC and the Amalgamated graph conjecture.
- Tardos (1996)

$$\bar{\mathrm{r}}(H \cap K) \leq 2\,\bar{\mathrm{r}}(H)\,\bar{\mathrm{r}}(K) - \bar{\mathrm{r}}(H) - \bar{\mathrm{r}}(K) + 1.$$

- ...

Results

- Dicks and Formanek (2001) SHNC for $\mathrm{r}(H) \leq 3$.
- Meakin and Weil (2002), independently Khan (2002), SHNC for H generated by positive words.
- Kent (2009)

$$\bar{\mathrm{r}}(H \cap K) \leq 2\,\bar{\mathrm{r}}(H)\,\bar{\mathrm{r}}(H) - \bar{\mathrm{r}}(H)\,\bar{\mathrm{r}}(\langle\, H \cup K\,\rangle).$$

- Many others investigated intersections of free groups including: Stallings, Gersten, Imrich, Nickolas, Louder, Servatius, Ivanov, Everett...

Mineyev's solution of the SHNC

In May 2011, Igor Mineyev put two preprints with two solutions of the SHNC. (Link)

[Min1] Igor Mineyev, *Submultiplicativity and the Hanna Neumann Conjecture*, Preprint, May 2011.

[Min2] Igor Mineyev, *Groups, Graphs, and the Hanna Neumann conjecture*, Preprint, May 2011.

Mineyev's original solution [Min1] involves Hilbert modules and Murray-von Neumann dimension, plus some graph theory.

People with some background in analysis may found [Min1] conceptually simpler that the proof we are about to present.

The SHNC is a particular corollary of a much general result of [Min1] about intersection of complexes, not just trees.

Dicks reduced Mineyev's proof from a 20-page argument to a based on Bass-Serre theory. Mineyev expanded it into a 7-page argument [Min2] now known as "Mineyev's second proof."

This talk is based on the 1-page proof of Dicks. (Link)

Trees

- Let T be Cayley graph of F with respect to $\{a, b\}$. Then T is tree where F acts freely.
- For each $G \leqslant F$, let T_G be the smallest G-subtree of T.
- The fundamental group of $G \backslash T_G$ is G. In particular, $\mathrm{r}(G) = \mathrm{r}(G \backslash T_G)$.
- For $H \leqslant G$, we have natural inclusions $T_H \to T_G$ that induces a maps

$$H \backslash T_H \to G \backslash T_G.$$

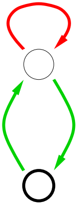
- There is an injective map

$$(H \cap K) \backslash T_{H \cap K} \to H \backslash T_H \times K \backslash T_K.$$

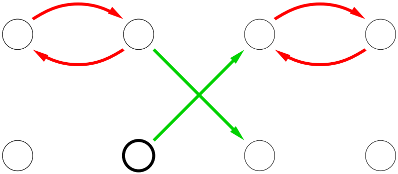
- We can understand $H \cap K$ by understanding the image of the previous map.

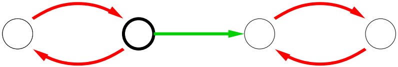
Example

- $H = \langle bab^{-1}, b^2 \rangle$, $K = \langle a^2, ba^2b^{-1} \rangle$



$H \backslash T_H$





$K \backslash T_K$

Two observations from the example

From the example we observe the following

- if H and K are finitely generable, so is $H \cap K$.
- Each connected component in the image corresponds, up to conjugation, to a subgroup $x^{-1}Hx \cap y^{-1}Ky$.
- We have a natural injective map

$$\bigcup_{x \in H\backslash F/K} (H^x \cap K)\backslash T_{H^x \cap K} \to H\backslash T_H \times K\backslash T_K.$$

and it follows $\sum_{x \in H\backslash F/K} \bar{\mathrm{r}}(H^x \cap K)$ is bounded.

Imagine trees...

- Suppose that for all $G \leqslant F$ we have a G-tree $\mathcal{T}_G$,
- such that, there are injective maps

$$(1) \qquad \bigcup_{x \in H \backslash F / K} (H^x \cap K) \backslash \mathcal{T}_{H^x \cap K} \to H \backslash \mathcal{T}_H \times K \backslash \mathcal{T}_K .$$

- and

$$\bar{\mathrm{r}}(G) = |G \backslash E\mathcal{T}_G| \quad (*).$$

Then the injectivity of (1) implies

$$\sum_{x \in H \backslash F / K} |(H^x \cap K) \backslash E\mathcal{T}_{H^x \cap K}| \leq |H \backslash E\mathcal{T}_H||K \backslash E\mathcal{T}_K|$$

and hence SHNC follows from (*).

Do not imagine

Theorem

Such trees exist.

Ordered trees

- Let T the Cayley graph of F wrt $\{a, b\}$. Then $VT = F$ and $ET = F \times \{(1, a), (1, b)\}$.
- We order ET lexicographically by
  - putting F-left-invariant order on F, (F is orderable)
  - putting any total order in $\{(1, a), (1, b)\}$
- The lexicographic order ET is F-left-invariant. We denote it by $<$.

Bridges and islands

Definition

- An edge $e \in ET_G$ is a G-bridge (or order-essential) if there exists some reduced bi-infinite path in T_G in which e is the $<$-largest edge.
- Let $B(G)$ denote the set of G-bridges of T_G. Note that $B(G)$ is a free G-set.
- A connected component of $T - B(G)$ is called a G-island (or a component of order-inessetial edges).
- Let $I(G)$ denote the components of G-islands. Note that $I(G)$ is a G-set.

For $G \leqslant F$, let $\mathcal{T}_G$ be the G-tree with $E\mathcal{T}_G = B(G)$ the set of G-bridges and $V\mathcal{T}_G = I(G)$ the set of G-islands.

These trees are inspired by the concept of leafages.

Exercise

Prove the SHNC

The pullback

- If $H \leqslant G$, then $T_H \subseteq T_G$.
- If e is an H-bridge, then e is a G-bridge. Then the inclusion $T_H \subseteq T_G$ induces an injective H-map $\mathcal{T}_H \to \mathcal{T}_G$, which induces a map $H \backslash \mathcal{T}_H \to G \backslash \mathcal{T}_G$.
- The injective map $\bigcup_{x \in H\backslash F/K} (H^x \cap K) \backslash T_{H^x \cap K} \to H \backslash T_H \times K \backslash T_K$ induces an injective map

$$\bigcup_{x \in H\backslash F/K} (H^x \cap K) \backslash \mathcal{T}_{H^x \cap K} \to H \backslash \mathcal{T}_H \times K \backslash \mathcal{T}_K.$$

- To prove SHNC we only have to show that $\bar{\mathrm{r}}(G) = |G \backslash B(G)|$.

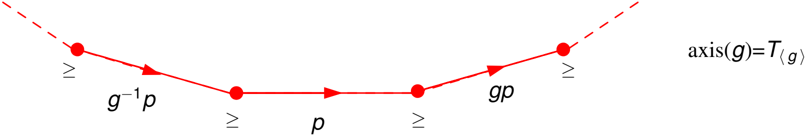
The rank 0 case

If $G = \{1\}$, T_G is empty and hence $B(G)$ is empty. Hence $\mathcal{T}_G$ is empty and

$$\bar{\mathrm{r}}(G) = 0 = |G \backslash B(G)|.$$

The rank 1 case

Let $g \in F - \{1\}$. Then T_G is just the axis of g.



If $G = \langle g \rangle$, there is no G-bridge

- T_G is the only bi-infinite path.
- Let v be any vertex in T_G, and p the path joining v and gv. Let e the $<$-largest edge in p.
- If $e > ge$, then $\cdots > \max(g^{-1}p) > \max(p) > \max(gp) > \ldots$ and there is no G-bridge.

In this case $\mathcal{T}_G =$ 

Computing rank

Before $\mathrm{r}(G) = \mathrm{r}(G\backslash T_G)$.

But now, G may not act freely on $\mathcal{T}_G$.

How $G\backslash\mathcal{T}_G$ is related to $\mathrm{r}(G)$?

Computing rank through graphs of groups

Euler characteristic of graphs of groups

Suppose that G is the fundamental group of a graph of groups $(Y, (G_v : v \in VY), (G_e : e \in EY))$ then

$$\chi(G) = \sum_{v \in VY} \chi(G_v) - \sum_{e \in EY} \chi(G_e)$$

Let $(Y, (G_i : i \in VY), (G_e : e \in EY))$ be the graph of groups of G associated to its action of $\mathcal{T}_G$, then G_i is a stabilizer of a G-island and $G_e = \{1\}$. Remember that $\chi(\{1\}) = 1$. We have,

$$\chi(G) = \sum_{i \in G \backslash I(G)} \chi(G_i) - |G \backslash B(G)|$$

The islands

$$\chi(G) = \sum_{i \in G \setminus I(G)} \chi(G_i) - |G \setminus B(G)|$$

Recall that

- $\bar{\mathrm{r}}(G) = \max\{0, -\chi(G)\}$.
- SHNC follows if we show $\bar{\mathrm{r}}(G) = |G \setminus B(G)| \quad (*)$
- If G is trivial, $(*)$ holds.
- If G is non-trivial, $\bar{\mathrm{r}}(G) = -\chi(G)$.
- If G is non-trivial, if $\sum_{i \in G \setminus I(G)} \chi(G_i) = 0$, $(*)$ follows.

We are going to show

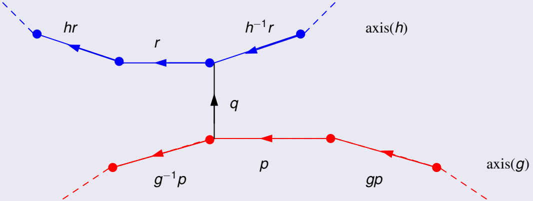
- $\chi(G_i) \geq 0$.
  Hence, $\bar{\mathrm{r}}(G) \leq -\chi(G) \leq |G \setminus B(G)|$, and if $\bar{\mathrm{r}}(G) = \infty$, $(*)$ holds.
- if G is finitely generable then $\chi(G_i) = 0$.
  Hence, $\bar{\mathrm{r}}(G) = -\chi(G) = |G \setminus B(G)|$, that is, $(*)$ holds.

Theorem

If $\mathrm{r}(G) \geq 2$ *then* $B(G)$ *is nonempty.*

Proof.

- Suppose that $g, h \in G$ generate a free group of rank 2.



- after replacing g and p with g^{-1} and $g^{-1}p^{-1}$ we can assume that the $<$-largest edge e of p satisfies $e > ge$
- then the reduced bi-infinite path $\cdots g^2p \cdot gp \cdot p \cdot q \cdot r \cdot hr \cdot hr^2 \cdots$ has $<$-largest edge in $p \cdot q \cdot r$.

□

Theorem

If G is f.g. and non-trivial, then the stabilizers of the islands are non-trivial.

Proof.

- Let I be an island, and v a vertex in I, and p and infinite ray in T_G starting at v.
- Since G is f.g, $N := |G \backslash ET_G| + 1 < \infty$.
- Let $e_1 < e_2 < e_3 < \cdots < e_s$ be the G-bridges at distance at most N from v.
- Draw a picture.
- After doing finite tail replacements, we obtain an infinite ray in T_G starting at v that do not cross any bridge at distance at most N from v.
- $|EI| > N - 1 = |G \backslash ET|$, and hence there is $e \in EI$ and $g \in G$ such that $ge \in EI$ and $ge \neq e$.
- Since $I(G)$ is a G-set, $g \in G_I$, therefore $G_I \neq \{1\}$.

□

Summarizing

$$\bigcup_{x\in H\backslash F/K}(H^x\cap K)\backslash\mathcal{T}_{H^x\cap K}\to H\backslash\mathcal{T}_H\times K\backslash\mathcal{T}_K.$$

- By construction.

$$\bar{\mathrm{r}}(G)=|G\backslash E\mathcal{T}_G| \quad (*)$$

- $(*)$ holds for $G=\{1\}$.
- By the first theorem, since the islands contain no bridges $\bar{\mathrm{r}}(G_I)=0$.
- Then $(*)$ holds for $\bar{\mathrm{r}}(G)=\infty$.
- By the second theorem, if $\mathrm{r}(G)\neq 0,\infty$ the islands have non-trivial stabilizers.
- Hence $\mathrm{r}(G_i)=1$ and then $\chi(G_i)=0$.
- By the Euler Characteristic formula for graph of groups $\bar{\mathrm{r}}(G)=|G\backslash E\mathcal{T}_G|$.

THE  CONJECTURE

IS TRUE.