

# Logical equations and types

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The aim of the theory introduced by B.I. Plotkin is to consider the interaction between algebraic structure of an algebra and its geometrical and logical properties.

- ▶ **Universal Algebra**
- ▶ **Algebraic Logic**
- ▶ **Algebraic Geometry**

The concept of *an LG-type* plays an exceptional role.  
In this concern the new logical invariants and logical relations  
of algebras were introduced:

- ▶ isotypeness,
- ▶ logically separable algebras,
- ▶ logically noetherian algebras,
- ▶ logically perfect algebras,
- ▶ ...

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## The main goal

- ▶ To give some basic notions of this theory.
- ▶ To show that the concept of an  $LG$ -type has deep ties with geometric properties of algebras.
- ▶ In particular, we shall show that  $LG$ -types appear naturally in the Galois correspondence between filters in the Halmos algebra of first order formulas over equalities and elementary sets in the corresponding affine space.

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# Algebraic geometry

## Classical algebraic geometry

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<b>Classical AG</b>
<i>Com</i> – $K$
$K$ - field
$K[X] = K[x_1, \dots, x_n],  X  = n$
$f(x_1, \dots, x_n) \in K[X]$
$f(x_1, \dots, x_n) \equiv 0$
$K^n$
$(a_1, \dots, a_n) \in K^n$
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Let us identify a point

$$(a_1, \dots, a_n) \in K^n$$

with the homomorphism

$$\mu = \mu_a : K[x_1, \dots, x_n] \rightarrow K$$

by  $\mu(x_i) = a_i$ . So,  $\mu = (a_1, \dots, a_n)$ . Then

$$K^n \cong \text{Hom}(K[X], K)$$

If  $f = f(x_1, \dots, x_n) \in K[X]$ , then

$$\mu(f) = f(\mu(x_1), \dots, \mu(x_n)) = f(a_1, \dots, a_n).$$

The point  $\mu$  is a *solution* of a polynomial  $f$  iff

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## Classical algebraic geometry

### Galois correspondence

Let  $T$  be a set of polynomials in  $K[x_1, \dots, x_n]$ .

polynomials  $T \rightarrow$  set of common zeros  $A = T'_K$

$T$

$\Downarrow$

$$A = \{(a_1, \dots, a_n) \in K^n \mid f(a_1, \dots, a_n) = 0, \text{ for all } f \in T\},$$

or

$$A = \{\mu \in \text{Hom}(K[X], K) \mid T \subset \text{Ker}(\mu)\}.$$

ideals in  $K[x_1, \dots, x_n] \rightarrow$  algebraic sets in  $K^n$

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Let  $A$  be an arbitrary set in the affine space  $K^n$ .

set  $A \rightarrow$  the set  $T = A'_K$  of all polynomials vanishing at  $A$

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$T = \{f \in K[x_1, \dots, x_n] \mid f(a_1, \dots, a_n) = 0, \text{ for all } (a_1, \dots, a_n) \in A\}$

or

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sets in  $K^n \rightarrow$  ideals in  $K[x_1, \dots, x_n]$ .

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$f(x_1, \dots, x_n) \equiv 0$
$K^n \cong Hom(K[X], K)$
$\mu = (a_1, \dots, a_n) \in K^n$
$f(a_1, \dots, a_n) = 0$ or $\mu(f) = 0$
$\mu$ is a solution of $f$
$\Leftrightarrow f \in Ker(\mu)$
algebraic sets $\Leftrightarrow$ $\Leftrightarrow$ closed ideals

# Universal algebraic geometry

Classical AG	Universal AG
$Com - K$	$\Theta$
$K$ - field	
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$Com - K$	$\Theta$
$K$ - field	$H$ - algebra in $\Theta$
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$K[X] = K[x_1, \dots, x_n],  X  = n$	$W(X)$ free algebra in $\Theta$
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$f(x_1, \dots, x_n) \in K[X]$	$w(x_1, \dots, x_n) \in W(X)$
$f(x_1, \dots, x_n) \equiv 0$	
$K^n \cong Hom(K[X], K)$	
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$f(x_1, \dots, x_n) \in K[X]$	$w(x_1, \dots, x_n) \in W(X)$
$f(x_1, \dots, x_n) \equiv 0$	$w \equiv w'$
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## Algebraic geometry over:

- ▶ the free metabelian group (V. Remeslennikov, R. Stohr, N. Romanovskii, E. Timoshenko);
- ▶ solvable groups (C.K. Gupta, A. Myasnikov, N. Romanovskii);
- ▶ commutative monoids with cancellation (P. Morar, A. Shevlyakov);
- ▶ Lie algebras (E. Daniyarova, I. Kazachkov, V. Remeslennikov, N. Romanovskii, I. Shestakov, R. Stohr).

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$f(a_1, \dots, a_n) = 0$ or $\mu(f) = 0$	$w(a_1, \dots, a_n) = w'(a_1, \dots, a_n)$
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$K$ - field	$H$ - algebra in $\Theta$
$K[X],  X  = n$	$W(X)$ free algebra in $\Theta$
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## Universal algebraic geometry

### Galois correspondence

$$T = \{w(x_1, \dots, x_n) \equiv w'(x_1, \dots, x_n) \mid w, w' \in W(X)\}$$



$$A = T'_H = \{\mu = (a_1, \dots, a_n) \in \text{Hom}(W(X), H) \mid \\ \mu(w) = \mu(w'), \text{ for all } w \equiv w' \in T\}$$

congruences on  $W(X)$



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# Logical geometry

## Tarski Problem

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*Is it true that two non-commutative free groups  $F_n$  and  $F_m$  with different  $n$  and  $m$  are elementary equivalent?*

Two groups  $G_1$  and  $G_2$  are elementary equivalent if  $Th(G_1) = Th(G_2)$ , where  $Th(G_i)$  is the set of all FOL sentences which hold true in the group  $G_i$ .

Theorem (O. Kharlampovich+A.Myasnikov; Z. Sela)

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## The algebra $\Phi$

Multi-sorted (infinite-sorted) algebra of formulas

$$\Phi = (\Phi(X_i), X_i \in \Gamma)$$

- $X^0$  is an infinite set of variables.
- $\Gamma$  is the system of all finite subset  $X_i$  in  $X^0$ .

Geometry	Logic
$X$ is a finite set	$X$ is an infinite set

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1.  $\Phi(X_i)$  is a boolean algebra with the operations  $\wedge, \vee, \neg$  over equalities of the form  $w \equiv w', w, w' \in W(X_i)$ ;
2. On  $\Phi(X_i)$  the operations  $\exists x_j$  is define for all  $x_j \in X_i$ ;
3. There are operations  $s_* = s_*^{X_i X_j}$  for various  $X_i, X_j \in \Gamma$ :

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**Warning.** *The existence of operations  $s_*$  breaks our intuition about structure of formulas in many-sorted case. If an element  $u$  belong to  $\Phi(X)$ , then we cannot represent explicitly the element  $u = s_*v$  from  $\Phi(X)$  in terms of equalities, connectives, and quantifiers in  $\Phi(X)$ . This means that we cannot trace the structure of an arbitrary element from  $\Phi(X)$ .*

## Example

$$s_*(x_1 \equiv x_2^3) \vee (x_2 \equiv x_3^3) \wedge s_*\left(\exists x_1(x_1 x_2 \equiv x_3)\right).$$

$$s : W(x_1, x_2, x_3) \rightarrow W(x_2, x_3),$$

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$K$ - field	$H \in \Theta$	$H \in \Theta$
$K[X],  X  = n$	$W(X)$ free algebra in $\Theta$	$W(X), \Phi(X)$
$f(x_1, \dots, x_n) \in K[X]$	$w(x_1, \dots, x_n) \in W(X)$	$w(x_1, \dots, x_n) \in W(X)$
$f_i(x_1, \dots, x_n) \equiv 0$	$w \equiv w'$	$w \equiv w', u \in \Phi(X)$
$K^n \cong Hom(K[X], K)$	$H^n \cong Hom(W(X), H)$	$H^n \cong Hom(W(X), H)$
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## Remark

*Really, we do not know how formulas from  $\Phi(X_i)$  look like. To by pass this difficulty the algebra  $Bool(W(X_i), H)$  is introduced. It is the algebra where the values (solutions) of the formulas are situated. And we "study" formulas using their values.*

## Algebra $(Bool(W(X_i), H), X_i \in \Gamma)$

- ▶  $Bool(W(X_i), H)$  is a boolean algebra of all subsets in  $Hom(W(X_i), H)$  with operations  $\cup, \cap, \neg$ .
- ▶ The operations  $\exists x$  are defined on  $Bool(W(X_i), H)$  for all  $x \in X_j$ .
- ▶ The operations

$$s_* : Bool(W(X_i), H) \rightarrow Bool(W(X_j), H)$$

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## The homomorphism $Val_H$

Algebraic approach:

$Val_H :$

$$(\Phi(X_i), X_i \in \Gamma) \longrightarrow (Bool(W(X_i), H), X_i \in \Gamma).$$

The homomorphism of multi-sorted algebras

$$\varphi : (A_i, i \in \Gamma) \rightarrow (B_i, i \in \Gamma)$$

has the form

$$\varphi = (\varphi_i, i \in \Gamma), \varphi_i : A_i \rightarrow B_i,$$

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$$Val_H^{X_i} : \Phi(X_i) \xrightarrow{hom} Bool(W(X_i), H).$$

$$Val_H^{X_i}(w \equiv w') = \{\mu : W(X_i) \rightarrow H \mid \mu(w) = \mu(w')\},$$

where  $w, w' \in W(X_i)$ .

Each  $Val_H^{X_i}$  preserves the operations  $\vee, \wedge, \neg, \exists x, x \in X_i, s_*$ ,  
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## The value (solution) of a formula

### Definition

*A point  $\mu \in \text{Hom}(W(X), H)$  is a solution of a formula  $u \in \Phi(X)$  if  $\mu$  belongs to the set  $\text{Val}_H^X(u)$ .*

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## Logical kernel of a point

$$\text{Ker}(\mu) = \{f \in K[X] \mid \mu \text{ is a solution of } f\}$$

### Definition

$$\text{LKer}(\mu) = \{u \in \Phi(X) \mid \mu \text{ is a solution (value) of } u\},$$

or

$$\text{LKer}(\mu) = \{u \in \Phi(X) \mid \mu \in \text{Val}_H^X(u)\}.$$

- ▶  $\text{Ker}(\mu)$  is a congruence on  $W(X)$ : a normal subgroup, if  $W(X)$  is a group; an ideal, if  $W(X)$  is an algebra, ... ).
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A subset  $F$  of a boolean algebra  $B$  is a filter if for every  $a_1, a_2 \in F$  and  $b \in B$  we have

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## Logical geometry

### Galois correspondence

Let  $T$  be a set of formulas in  $\Phi(X)$ .

$$\begin{array}{c} T \\ \Downarrow \\ A = T_H^L = \bigcap_{u \in T} \text{Val}_H^X(u). \end{array}$$

The set  $A$  is called *an elementary set (definable sets)*.

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## Logically geometrical type (LG-type)

### Definition

*Every ultrafilter  $T$  in the algebra  $\Phi(X)$  is called an  $X$ -LG-type.*

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*An LG-type  $T$  is realizable in  $H$ , if there is a point  $\mu \in \text{Hom}(W(X), H)$  such that  $T = \text{LKer}(\mu)$ .*

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## Model theoretical type

- ▶  $M$  is a set,  $C \subseteq M$ .
- ▶  $\mathbb{L}_C$  is a language with the set of constant  $C$  and  $\mathbb{M} = (\mathbb{L}_C, M)$  is an  $\mathbb{L}_C$ -structure.
- ▶  $Th_C(\mathbb{M})$  is the set of all  $\mathbb{L}_C$ -sentences true in  $\mathbb{M}$ , that is the  $\mathbb{L}_C$ -theory (*elementary theory*) of the model  $\mathbb{M}$ .

## Model theoretical type

### Definition

Let  $P = \{u_i(x_1, \dots, x_n)\}$  be the set of  $\mathbb{L}_C$ -formulas in free variables  $x_1, \dots, x_n$ . We call  $P$  a model theoretical  $n$ -type if  $P \cup Th_C(\mathbb{M})$  is satisfiable.

### Definition

$P$  is a complete  $n$ -type if  $u \in P$  or  $\neg u \in P$  for all  $\mathbb{L}_C$ -formulas  $u$  with free variables from  $x_1, \dots, x_n$ .

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An  $n$ -type  $P$  is realizable on  $(a_1, \dots, a_n) \in M^n$  if  $u_i(a_1, \dots, a_n)$  is true in  $\mathbb{M}$  for all  $u_i \in P$ .

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## LG-isotyped algebras

$S^X(H)$  is the set of all LG-types in  $\Phi(X)$  which are realizable in  $H$ .

### Definition

*Two algebras  $H_1$  and  $H_2$  are called LG-isotyped if the sets of their LG-types coincide:*

$$S^X(H_1) = S^X(H_2),$$

*for every  $X \in \Gamma$ .*

## MT-isotyped algebras

$S_n(H)$  is the set of all model theoretical  $n$ -types which are realizable in  $H$ .

### Definition

*Two algebras  $H_1$  and  $H_2$  are called MT-isotyped if the sets of their MT-types coincide:*

$$S_n(H_1) = S_n(H_2),$$

*for every  $n \in \mathbb{N}$ .*

Theorem (B. Plotkin, G. Zhitomirskii, 2011)

*Two algebras are LG-isotyped if and only if they are MT-isotyped.*