

The lattice of subgroups of a free group

Enric Ventura

Departament de Matemàtica Aplicada III
Universitat Politècnica de Catalunya

GAGTA-5 mini-course, 2011

July 5th, 2011.

Outline

- 1 Notation
- 2 Automata
- 3 Schreier graphs
- 4 First algebraic applications
- 5 Finite index subgroups
- 6 Intersections of subgroups
- 7 Fringe and algebraic extensions
- 8 The pro- \mathcal{V} topology
- 9 Fixed points

Outline

- 1 Notation
- 2 Automata
- 3 Schreier graphs
- 4 First algebraic applications
- 5 Finite index subgroups
- 6 Intersections of subgroups
- 7 Fringe and algebraic extensions
- 8 The pro- \mathcal{V} topology
- 9 Fixed points

Free monoid

Definition

- $A = \{a_1, \dots, a_r\}$ is a finite alphabet.
- A^* is the free monoid on A .
- A language is a subset $L \subseteq A^*$.
- An involutive alphabet $\tilde{A} = \{a_1, \dots, a_r, a_1^{-1}, \dots, a_r^{-1}\}$.
- Reduced words; reduction $\sim; R(A)$.
- Formal word definitions

$$(a^{-1})^{-1} = a,$$

$$(a_{i_1}^{\epsilon_1} \dots a_{i_k}^{\epsilon_k})^{-1} = a_{i_k}^{-\epsilon_k} \dots a_{i_1}^{-\epsilon_1}.$$

Free group

Definition

The *free group* on A , $F(A) = \tilde{A}^* / \sim$.

Lemma

For every $w \in \tilde{A}^*$, there is a unique $\bar{u} \in R(A)$, s.t. $u =_{F(A)} \bar{u}$.

Lemma

$F(A)$ is a quotient of \tilde{A}^* . The projection is denoted π :

$$\begin{aligned} \pi: \tilde{A}^* &\rightarrow F(A) \\ u &\mapsto [u] = [\bar{u}]. \end{aligned}$$

Definition

For a subgroup $H \leq F(A)$, define $\bar{H} = \{\bar{u} \mid u \in H\} \subseteq \tilde{A}^*$.

Outline

- 1 Notation
- 2 Automata**
- 3 Schreier graphs
- 4 First algebraic applications
- 5 Finite index subgroups
- 6 Intersections of subgroups
- 7 Fringe and algebraic extensions
- 8 The pro- \mathcal{V} topology
- 9 Fixed points

Paths

From now on, *all* automata we consider will be involutive.

Definition

Let \mathcal{A} be an A -automata.

- A *path* γ in \mathcal{A} ,
- the *label* of a path γ , $\text{label}(\gamma) \in \tilde{A}^*$,
- *reduced* path,
- notation: $p \xrightarrow{u} q$ means a path from p to q with label $u \in \tilde{A}^*$.

Lemma

Let $p \xrightarrow{u} q$ be a path in \mathcal{A} . If u is reduced then $p \xrightarrow{u} q$ is reduced.
The convers is not true.

Trimness

Definition

The *language* of an A -automata \mathcal{A} , is

$$L(\mathcal{A}) = \{u \in \tilde{A}^* \mid \exists q_0 \xrightarrow{u} q_0\} \subseteq \tilde{A}^*.$$

Definition

An A -automata \mathcal{A} is *trim* if it has no vertices of degree 1 except maybe the basepoint.

Lemma

If \mathcal{A} is trim then $\forall q \neq q_0$ there exists a reduced path $q_0 \rightarrow q \rightarrow q_0$.

Determinism

Definition

An A -automata \mathcal{A} is *deterministic* if $(p, a, q) \in E$ and $(p, a, q') \in E$ imply $q = q'$.

Lemma

Let \mathcal{A} be a deterministic A -automaton. We have,

- i) if $p \xrightarrow{u} q$ is reduced then u is reduced,
- ii) if $\exists p \xrightarrow{u} q, \exists p \xrightarrow{u} q'$ then $q = q'$,
- iii) if $\exists p \xrightarrow{u} q, \exists p' \xrightarrow{u} q$ then $p = p'$.
- iv) if $\exists p \xrightarrow{uvw^{-1}w} q$, then $\exists p \xrightarrow{uw} q$.

Morphisms

Definition

Let $\mathcal{A} = (V, E, q_0)$ and $\mathcal{A}' = (V', E', q'_0)$ be two A -automata. A *morphism* $\mathcal{A} \rightarrow \mathcal{A}'$ is a map $\varphi: V \rightarrow V'$ such that $q_0\varphi = q'_0$ and

$$(p, a, q) \in E \Rightarrow (p\varphi, a, q\varphi) \in E'.$$

Proposition

Let $\mathcal{A} = (V, E, q_0)$ and $\mathcal{A}' = (V', E', q'_0)$ be two A -automata, \mathcal{A}' deterministic. Then,

$$L(\mathcal{A}) \subseteq L(\mathcal{A}') \Leftrightarrow \exists \text{ morphism } \varphi: \mathcal{A} \rightarrow \mathcal{A}'.$$

In this case, φ is unique.

This proof will be repeated with variations later.

Outline

- 1 Notation
- 2 Automata
- 3 Schreier graphs**
- 4 First algebraic applications
- 5 Finite index subgroups
- 6 Intersections of subgroups
- 7 Fringe and algebraic extensions
- 8 The pro- \mathcal{V} topology
- 9 Fixed points

Stallings folding

We want to *make* $\mathcal{F}(W)$ deterministic.

Definition

Let \mathcal{A} be an A -automaton, and suppose $e = (p, a, q)$ and $e' = (p, a, q')$ are two different edges (so, $q \neq q'$).

Consider $\mathcal{L} = \{\{f, f'\} \neq \{e, e'\} \mid f = (p', b, q), f' = (p', b, q') \text{ for some } p' \in V, b \in A\}$.

Consider the automata \mathcal{A}' to be \mathcal{A} identifying $q = q'$ (and so, $e = e'$, and $f = f'$ for every $\{f, f'\} \in \mathcal{L}$, if any). We define $\mathcal{A} \rightsquigarrow \mathcal{A}'$ to be a *Stallings folding*.

The number $\ell = |\mathcal{L}| \geq 0$ is called the *lost* of the folding. A folding is called *critical* when it has positive lost.

Stallings folding

Lemma

Let $\mathcal{A} \rightsquigarrow \mathcal{A}'$ to be a Stallings folding. Then

- i) If, in \mathcal{A} , $|\{a \in \tilde{A} \mid q \xrightarrow{a}\}| \geq 2 \forall q \neq q_0$, then the same is true in \mathcal{A}' .
- ii) \exists a morphism $\varphi: \mathcal{A} \rightarrow \mathcal{A}'$ (and so, $L(\mathcal{A}) \subseteq L(\mathcal{A}')$).
- iii) $L(\mathcal{A})\pi = L(\mathcal{A}')\pi$.

Corollary

Let $W \subset R(A)$, $|W| < \infty$, and consider a sequence of foldings $\mathcal{F}(W) \rightsquigarrow \dots \rightsquigarrow \mathcal{A}$ to a deterministic \mathcal{A} . Then,

- i) \mathcal{A} is deterministic and trim,
- ii) $L(\mathcal{A})\pi = H = \langle W \rangle \leq F(A)$.
- iii) $\overline{H} \subseteq L(\mathcal{A}) \subseteq H\pi^{-1}$.

Stallings folding

Lemma

Let $\mathcal{A} \rightsquigarrow \mathcal{A}'$ to be a Stallings folding. Then the natural map

$$\begin{aligned} \pi\mathcal{A}(q_0, q_0) &\rightarrow \pi\mathcal{A}'(q_0, q_0) \\ \gamma &\mapsto \gamma' = \gamma\varphi \text{ (+ canc. f.e.'s),} \end{aligned}$$

satisfies

- i) $label(\gamma)\pi = label(\gamma')\pi$,
- ii) *in the non-critical case, it is injective.*

Corollary

If all foldings in $\mathcal{F}(W) \rightsquigarrow \dots \rightsquigarrow \mathcal{A}$ are non-critical, then the above map, $\pi\mathcal{F}(W)(q_0, q_0) \rightarrow \pi\mathcal{A}(q_0, q_0)$, $\gamma \mapsto \gamma'$, is injective.

Membership problem

Theorem

The membership problem is solvable in $F(A)$: given $g, h_1, \dots, h_n \in F(A)$, one can decide whether $g \in H = \langle h_1, \dots, h_n \rangle$.

- (1) *can assume $W = \{h_1, \dots, h_n\} \subseteq R(A)$;*
- (2) *draw the flower automaton $\mathcal{F}(W)$;*
- (3) *apply an arbitrary sequence of foldings until a deterministic automaton $\mathcal{F}(W) \rightsquigarrow \dots \rightsquigarrow \mathcal{A}$;*
- (4) *start reading \bar{g} as (the label of) a path in \mathcal{A} , from q_0 ;*
- (5) *if not possible then $g \notin H$;*
- (6) *if possible (so, in a unique way) but as an open path then $g \notin H$;*
- (7) *if possible as a closed path at q_0 , then $g \in H$. □*

Independence of the process

In a sequence of Stallings foldings, $\mathcal{F}(W) \rightsquigarrow \dots \rightsquigarrow \mathcal{A}$, the result *will not* depend on the process, and even on W , but *only on* the subgroup $H = \langle W \rangle \leq F(A)$.

Theorem

\mathcal{A} depends only on $H = \langle W \rangle$, and is called the Schreier graph, $\Gamma(H)$.

Proposition

Let $H \leq_{f.g} F(A)$, choose a finite set of generators W , $\langle W \rangle = H$, and let $\mathcal{F}(W) \rightsquigarrow \dots \rightsquigarrow \mathcal{A}$ be an arbitrary sequence of Stallings foldings, with \mathcal{A} deterministic. Then,

$$L(\mathcal{A}) = \bigcap_{\mathcal{B}} L(\mathcal{B}),$$

where \mathcal{B} runs over all possible automata deterministic, trim, and such that $\overline{H} \subseteq L(\mathcal{B}) \subseteq \tilde{A}^*$.

Outline

- 1 Notation
- 2 Automata
- 3 Schreier graphs
- 4 First algebraic applications**
- 5 Finite index subgroups
- 6 Intersections of subgroups
- 7 Fringe and algebraic extensions
- 8 The pro- \mathcal{V} topology
- 9 Fixed points

Nielsen-Schreier Theorem

Let \mathcal{A} be an A -automata deterministic and trim, and let $H = L(\mathcal{A})\pi \leq_{f.g.} F(A)$.

Take a maximal tree T in \mathcal{A} and for every $e \in EA \setminus ET$ take

$$h_e = \text{label}(T[q_0, \iota e]eT[\tau e, q_0])\pi \in H.$$

Proposition

$\{h_e \mid e \in EA \setminus ET\}$ is a free basis for H .

Theorem (Nielsen)

Every finitely generated subgroup of a free group is free.

Theorem (Schreier)

Every subgroup of a free group is free.

Outline

- 1 Notation
- 2 Automata
- 3 Schreier graphs
- 4 First algebraic applications
- 5 Finite index subgroups**
- 6 Intersections of subgroups
- 7 Fringe and algebraic extensions
- 8 The pro- \mathcal{V} topology
- 9 Fixed points

The full Schreier graph

Definition

An A -automata \mathcal{A} is **complete** if every vertex has an edge going in and an edge going out with each label.

Definition

For $H \leq_{f.g.} F(A)$, define $\tilde{\Gamma}(H)$ to be $\Gamma(H)$ with infinite trees attached in order to make it complete.

Observation

- i) $\Gamma(H)$ is complete $\Leftrightarrow \tilde{\Gamma}(H) = \Gamma(H)$.
- ii) $\forall u \in \tilde{A}^*, \forall q \in V, \exists q \xrightarrow{u}$ in $\tilde{\Gamma}(H)$.
- iii) $L(\tilde{\Gamma}(H)) = H\pi^{-1}$.

Finite index subgroups

Proposition

Let $H \leq_{f.g.} F(A)$ and let T be a maximal tree in $\tilde{\Gamma}(H)$. Then,

$$\begin{aligned} \varphi: V\tilde{\Gamma}(H) &\rightarrow H \setminus F(A) \\ v &\mapsto H \cdot l_v \end{aligned}$$

is bijective, where $l_v = \text{label}(T[q_0, v])\pi$.

Corollary

Let $H \leq_{f.g.} F(A)$. Then, $H \leq_{f.i.} F(A) \Leftrightarrow \Gamma(H)$ is complete. In this case, $[F(A) : H] = |V\Gamma(H)|$.

Corollary (Schreier index formula)

Every $H \leq_{f.i.} F(A)$ is finitely generated and $r(H) - 1 = [F(A) : H](|A| - 1)$.

Finite index subgroups

Corollary

There is an algorithm which, given $h_1, \dots, h_n \in F(A)$, decides whether $H = \langle h_1, \dots, h_n \rangle$ is of finite index in $F(A)$ and, in this case, computes the index and a set of coset representatives.

Corollary

There is an algorithm which, given $h_1, \dots, h_n, k_1, \dots, k_m \in F(A)$, decides whether $\langle h_1, \dots, h_n \rangle = H \leq_{f.i.} K = \langle k_1, \dots, k_m \rangle$ and, in this case, computes the index and a set of coset representatives.

M. Hall's theorem

Definition

Let $H \leq K \leq F(A)$. We say that H is a *free factor* of K , denoted $H \leq_{f.f.} K$, if it is possible to extend a basis of H to a basis of K .

Lemma

For $H \leq_{f.g.} K \leq_{f.g.} F(A)$, if $\Gamma(H)$ is a subautomaton of $\Gamma(K)$ then $H \leq_{f.f.} K$. The convers is not true.

Theorem (M. Hall)

For every $H \leq_{f.g.} F(A)$ there exists $K \leq_{f.i.} F(A)$ such that $H \leq_{f.f.} K \leq_{f.i.} F(A)$.

Residual finiteness

Definition

A group G is said to be *residually finite* if $\forall 1 \neq g \in G$ there exists a finite quotient G/H where $1 \neq \bar{g} \in G/H$.

Theorem

Free groups are residually finite.

Theorem

Free groups are residually p , for every prime p .

Outline

- 1 Notation
- 2 Automata
- 3 Schreier graphs
- 4 First algebraic applications
- 5 Finite index subgroups
- 6 Intersections of subgroups**
- 7 Fringe and algebraic extensions
- 8 The pro- \mathcal{V} topology
- 9 Fixed points

Howson property

Definition

A group G satisfies the *Howson property* if the intersection of two finitely generated subgroups is again finitely generated.

Theorem

Free groups satisfy the Howson property.

Theorem

There is an algorithm which, given $h_1, \dots, h_n, k_1, \dots, k_m \in F(A)$, computes a basis of $H \cap K$, where $H = \langle h_1, \dots, h_n \rangle$ and $K = \langle k_1, \dots, k_m \rangle$.

Hanna Neumann "Conjecture"

Definition

Define the *reduced rank* of $H \leq F(A)$ as $\tilde{r}(H) = \max\{0, r(H) - 1\}$.

Theorem

For $H, K \leq_{f.g.} F(A)$, $\tilde{r}(H \cap K) \leq 2\tilde{r}(H)\tilde{r}(K)$.

Hanna Neumann "Conjecture"

For $H, K \leq_{f.g.} F(A)$, $\tilde{r}(H \cap K) \leq \tilde{r}(H)\tilde{r}(K)$.

Theorem (Mineyev, (simpl. Dicks))

For $H, K \leq_{f.g.} F(A)$, $\tilde{r}(H \cap K) \leq \tilde{r}(H)\tilde{r}(K)$.

Malnormality

Definition

A subgroup of a group $H \leq G$ is *malnormal* if $H^g \cap H = 1$ for all $g \notin H$.

Proposition

There is an algorithm which, given $h_1, \dots, h_n \in F(A)$, decides whether $H = \langle h_1, \dots, h_n \rangle$ is malnormal in $F(A)$.

Observation

For $H, K \leq_{f.g.} F(A)$, the collection of intersections $H^u \cap K^v$, moving $u, v \in F(A)$, takes only finitely many values, up to conjugacy.

Outline

- 1 Notation
- 2 Automata
- 3 Schreier graphs
- 4 First algebraic applications
- 5 Finite index subgroups
- 6 Intersections of subgroups
- 7 Fringe and algebraic extensions**
- 8 The pro- \mathcal{V} topology
- 9 Fixed points

Motivation

- In basic linear algebra:

$$U \leq V \leq K^n \Rightarrow V = U \oplus L.$$

- In \mathbb{Z}^n , the analog is **almost true**:

$$U \leq V \leq \mathbb{Z}^n \Rightarrow \exists U \leq_{fi} U' \leq V \text{ s.t. } V = U' \oplus L.$$

- In $F(A)$, the analog is ...

far from true because $H \leq K \not\Rightarrow r(H) \leq r(K) \dots$

Motivation

- In basic linear algebra:

$$U \leq V \leq K^n \Rightarrow V = U \oplus L.$$

- In \mathbb{Z}^n , the analog is **almost true**:

$$U \leq V \leq \mathbb{Z}^n \Rightarrow \exists U \leq_{fi} U' \leq V \text{ s.t. } V = U' \oplus L.$$

- In $F(A)$, the analog is ...

almost true again, ... in the sense of Takahasi.

Algebraic and transcendental elements

Mimicking field theory...

Definition

Let $H \leq F(A)$ and $w \in F(A)$. We say that w is

- algebraic over H if $\exists 1 \neq e_H(x) \in H * \langle x \rangle$ such that $e_H(w) = 1$;
- transcendental over H otherwise.

Observation

w is transcendental over $H \iff \langle H, w \rangle \simeq H * \langle w \rangle$
 $\iff H$ is contained in a proper f.f. of $\langle H, w \rangle$.

Problem

w_1, w_2 algebraic over $H \not\Rightarrow w_1 w_2$ algebraic over H .

$H = \langle a, \bar{a}bab, \bar{c}ac \rangle \leq \langle a, b, c \rangle$, and $w_1 = b, w_2 = \bar{c}$.

Takahasi's Theorem

Theorem (Takahasi, 1951)

For every $H \leq_{fg} F(A)$, the set of algebraic extensions, denoted $\mathcal{AE}(H)$, is finite.

- Original proof by Takahasi was combinatorial and technical,
- Modern proof, using Schreier automata, is **much simpler**, and due independently to Ventura (1997), Margolis-Sapir-Weil (2001) and Kapovich-Miasnikov (2002).
- Additionally, $\mathcal{AE}(H)$ is **computable**.

Fringe of a subgroup

Definition

Let \mathcal{A} be a deterministic and trim A -automata, and let \sim an eq. rel. on $V\mathcal{A}$. We denote by \mathcal{A}/\sim the new (deterministic and trim) A -automata resulting from identifying the vertices according to \sim , plus foldings.

Definition

The **fringe** of \mathcal{A} is the (finite) collection of A -automata of the form \mathcal{A}/\sim .

Definition

Let $H \leq_{fg} F(A)$. The **fringe** of H is $\mathcal{O}(H) = \{L(\Gamma(H)/\sim)\pi \mid \sim \text{ eq. rel. on } V\mathcal{A}\}$.

Computing $\mathcal{AE}(H)$

Corollary

For $H \leq_{fg} F(A)$, $\mathcal{AE}(H)$ is computable.

- 1) *Compute $\Gamma(H)$,*
- 2) *Compute $\Gamma(H)/\sim$ for **all eq. rel.** \sim of $V\Gamma(H)$,*
- 3) *Compute $\mathcal{O}(H)$,*
- 4) *Clean $\mathcal{O}(H)$ by detecting all pairs $K_1, K_2 \in \mathcal{O}(H)$ such that $K_1 \leq_{ff} K_2$ and deleting K_2 .*
- 5) *The resulting set is $\mathcal{AE}(H)$. \square*

For the cleaning step we need:

Deciding free-factoriness

Proposition

Given $H, K \leq F(A)$, it is algorithmically decidable whether $H \leq_{ff} K$ or not.

Proved by:

- Whitehead 1930's (classical and exponential),
- Silva-Weil 2006 (faster but still exponential),
- Roig-Ventura-Weil 2007 (variation of Whitehead algorithm in polynomial time),
- Puder 2011 (graphical argument).

Outline

- 1 Notation
- 2 Automata
- 3 Schreier graphs
- 4 First algebraic applications
- 5 Finite index subgroups
- 6 Intersections of subgroups
- 7 Fringe and algebraic extensions
- 8 The pro- \mathcal{V} topology**
- 9 Fixed points

Pseudo-varieties

Definition

A *pseudo-variety* of groups \mathcal{V} is a class of finite groups closed under taking subgroups, quotients and finite direct products.

- i) \mathcal{G} = all finite groups,
- ii) \mathcal{G}_p = all finite p -groups,
- iii) \mathcal{G}_{nil} = all finite nilpotent groups,
- iv) \mathcal{G}_{sol} = all finite soluble groups,
- v) \mathcal{G}_{ab} = all finite abelian groups,
- vi) for a finite group V , $[V]$ = all quotients of subgroups of V^k , $k \geq 1$.
- vii) ...

Definition

\mathcal{V} is *extension-closed* if $V \triangleleft W$ with $V, W/V \in \mathcal{V}$ imply $W \in \mathcal{V}$.

The pro- \mathcal{V} topology

Definition

Let G be a group, and \mathcal{V} be a pseudo-variety of finite groups. The *pro- \mathcal{V} topology on G* can be defined in several equivalent ways:

- i) it is the smallest topology making all the morphisms from G into all $V \in \mathcal{V}$ (with the discrete topology) continuous,
- ii) a basis of open sets is given by $\varphi^{-1}(x)$, for all morphism $\varphi: G \rightarrow V \in \mathcal{V}$,
- iii) the normal (finite index) subgroups $K \trianglelefteq G$ such that $G/K \in \mathcal{V}$ form a basis of neighborhoods of 1,
- iv) it is the topology given by the pseudo-ultra-metric $d(x, y) = 2^{-r(x,y)}$, where $r(x, y) = \min\{|V| \mid V \in \mathcal{V} \text{ and separates } x \text{ and } y\}$.

Observation

This topology is Hausdorff $\iff d$ is an ultra-metric $\iff G$ is residually- \mathcal{V} .

The \mathcal{V} -closure

Proposition

Let G be a group equipped with the pro- \mathcal{V} topology, and let $H \leq G$. Then, TFAE:

- (a) H is open
- (b) H is clopen (i.e. open and closed)
- (c) $H \leq_{fi} G$ and $G/H_G \in \mathcal{V}$.

Furthermore,

$$ch_{\mathcal{V}}(H) = \bigcap_{H \leq K, \text{ open}} K = \bigcap_{\varphi: G \rightarrow V \in \mathcal{V}} \varphi^{-1}(\varphi(H)).$$

Outline

- 1 Notation
- 2 Automata
- 3 Schreier graphs
- 4 First algebraic applications
- 5 Finite index subgroups
- 6 Intersections of subgroups
- 7 Fringe and algebraic extensions
- 8 The pro- \mathcal{V} topology
- 9 Fixed points**

Fixed subgroups are complicated

$$\begin{aligned} \phi: F_3 &\rightarrow F_3 \\ a &\mapsto a \\ b &\mapsto ba \\ c &\mapsto ca^2 \end{aligned}$$

$$\text{Fix } \phi = \langle a, bab^{-1}, cac^{-1} \rangle$$

$$\begin{aligned} \varphi: F_4 &\rightarrow F_4 \\ a &\mapsto dac \\ b &\mapsto c^{-1}a^{-1}d^{-1}ac \\ c &\mapsto c^{-1}a^{-1}b^{-1}ac \\ d &\mapsto c^{-1}a^{-1}bc \end{aligned}$$

$$\text{Fix } \varphi = \langle w \rangle, \text{ where...}$$

$$w = c^{-1}a^{-1}bd^{-1}c^{-1}a^{-1}d^{-1}ad^{-1}c^{-1}b^{-1}acdada c d c d b c d a^{-1}a^{-1}d^{-1}a^{-1}d^{-1}c^{-1}a^{-1}d^{-1}c^{-1}b^{-1}d^{-1}c^{-1}d^{-1}c^{-1}daabcdaccdb^{-1}a^{-1}.$$

Inertia

Definition

A subgroup $H \leq F_n$ is called *inert* if $r(H \cap K) \leq r(K)$ for every $K \leq F_n$.

Theorem (Dicks-V, 96)

Let $G \subseteq \text{Mon}(F_n)$ be an arbitrary set of monomorphisms of F_n . Then, $\text{Fix}(G)$ is inert; in particular, $r(\text{Fix}(G)) \leq n$.

Theorem (Bergman, 99)

Let $G \subseteq \text{End}(F_n)$ be an arbitrary set of endomorphisms of F_n . Then, $r(\text{Fix}(G)) \leq n$.

Conjecture (V.)

Let $\phi \in \text{End}(F_n)$. Then $\text{Fix}(\phi)$ is inert.

The four families

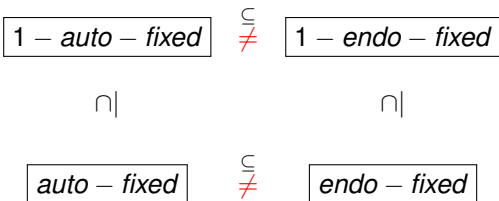
Definition

A subgroup $H \leq F_n$ is said to be

- *1-auto-fixed* if $H = \text{Fix}(\phi)$ for some $\phi \in \text{Aut}(F_n)$,
- *1-endo-fixed* if $H = \text{Fix}(\phi)$ for some $\phi \in \text{End}(F_n)$,
- *auto-fixed* if $H = \text{Fix}(S)$ for some $S \subseteq \text{Aut}(F_n)$,
- *endo-fixed* if $H = \text{Fix}(S)$ for some $S \subseteq \text{End}(F_n)$,

Easy to see that 1-mono-fixed = 1-auto-fixed.

Relations between them



Example (Martino-V., 03; Ciobanu-Dicks, 06)

Let $F_3 = \langle a, b, c \rangle$ and $H = \langle b, \text{c}acbab^{-1}c^{-1} \rangle \leq F_3$. Then, $H = \text{Fix}(a \mapsto 1, b \mapsto b, c \mapsto \text{c}acbab^{-1}c^{-1})$, but H is **NOT** the fixed subgroup of any set of automorphism of F_3 .

It is true up to free factors

Theorem (Martino-V., 00)

Let $S \subseteq \text{End}(F_n)$. Then, $\exists \phi \in \langle S \rangle$ such that $\text{Fix}(S) \leq_{\text{ff}} \text{Fix}(\phi)$.

However... free factors of 1-endo-fixed (1-auto-fixed) subgroups need not be even endo-fixed (auto-fixed).

Compression

Definition

A subgroup $H \leq F(A)$ is *compressed* when $r(H) \leq r(K)$ for every $H \leq K \leq F(A)$.

Observation

H inert $\Rightarrow H$ compressed.

Is every compressed subgroup, inert?

Proposition

There is an algorithm which, given $H \leq_{fg} F(A)$, decides whether H is compressed.

