The lattice of subgroups of a free group

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In this mini-course we will explore the lattice of subgroups of a free group via the so-called *Stallings graphs*. In 1983, J. Stallings made explicit in [5] a collection of ideas that were implicitly known to (and used by) other authors much before, like Schreier, Serre, Neumann, Dicks, etc (see, for example, [2], [3], [4]). Stallings' seminal paper was the beginning of a much more systematic and extensive use of these techniques, which soon proved to be much better and efficient than the classical combinatorial ones, in order to understand and investigate the properties of the lattice of finitely generated subgroups of a given free group. These graphical techniques happen to be easier to understand, more powerful, and more algorithmic friendly than the combinatorial ones: compare, for example, the graphical proof of Nielsen result that a finitely generated subgroup of a free group is free, with Nielsen's original one.

The key point is to think the free group F_r of rank r as the fundamental group of the bouquet with r petals R_r (i.e. the graph with only one vertex and r edges). Then, general covering theory establishes a bijection between subgroups of F_r and coverings of R_r; these coverings happen to be precisely a certain type of graphs called *Stallings graphs* (described by a very easy and clear conditions); the finite generation of the subgroup correspond to a certain finiteness condition on the Stallings graph; and the bijection is algorithmic friendly, in the sense that there are easy and fast algorithms to both built the Stallings graph of a given subgroup, and to read the subgroup from a given Stallings graph. Once this bijection is established and understood, a lot of algebraic problems about the lattice of subgroups of F_r translate into graphical problems involving Stallings graphs. In many occasions, this last problem can be solved using purely graphical techniques, and the solution pulled back into a solution to the original algebraic problem about subgroups.

Along the mini-course we shall build the theory of Stallings graphs, but following an alternative approach using finite inverse automata and language theory. This point of view makes the theory even easier and more elementary, specially for the people not familiar with covering theory: although the topological ideas from covering theory are underlying in many arguments, they will not be used all along the course.

The first hours of the mini-course will be dedicated to establish the details of the above mentioned Stallings graphs and bijection with finitely generated subgroups of F_r. The rest of the course will be dedicated to apply these techniques to solve several algorithmic problems about the lattice of subgroups of F_r: membership problem, conjugacy problem, determination of finite index, Hall's theorem, Howson property, computing intersections, etc. We will also dedicate a couple of hours to the notion of algebraic extension among subgroups (already present in the old paper [6]), how can it be studied using Stallings graphs, and to obtain several interesting algebraic applications, following [1].

References

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[3] C. Sims, *"Computation with finitely presented groups"*, Cambridge University Press, 1994.

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[5] J. R. Stallings, *"Topology of finite graphs"*, Inventiones Math. 71 (1983), 551–565.

[6] M. Takahasi, *"Note on chain conditions in free groups"*, Osaka J. Math., 3(2):221–225, 1951.