# Equations over free groups and Makanin-Razborov diagrams

Cornelius Reinfeldt

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This mini-course will try to cover the basic concepts of systems of equations over free groups and ways to solve them. Given a finite tuple of variables $X = (x_1, x_2, \ldots, x_n)$, a system of equations $\Sigma$ (without constants) over a group $G$ in the variables $x_1, \ldots, x_n$ is a (finite or infinite) set of words

$$\Sigma = \left\{ \begin{array}{c} w_1(X) \\ w_2(X) \\ \vdots \end{array} \right\},$$

where each $w_i$ is a word in the letters $X \cup X^{-1}$.

A solution of the system is a tuple $S = (g_1, \ldots, g_n) \in G^n$ of elements of $G$ such that

$$w_i(S) =_G 1 \qquad \text{for all } i,$$

i.e. that replacing each occuring variable $x_j^{\pm 1}$ in $w_i$ by $g_j^{\pm 1}$ yields a word which represents the identity in $G$.

It turns out that in general, it is by no means easy to decide whether or not a system of equations has a solution, never mind to describe the entire solution set. In the case where $G$ is free, however, these questions have positive answers. At first, G.S. Makanin ([M1]) presented an algorithm which decides whether or not a system of equations in a free group has a solution. The main tool of this work is a rewriting process which translates a system of equations into a finite set of *generalized equations*, a certain type of systems of equations in a free semigroup. This approach was adapted and refined by A. Razborov ([Ra]), who provided a description of the whole solution set of a given system of equations, the *Makanin-Razborov diagram.*

The first goal of this mini-course will be to get a solid understanding of Makanin's rewriting process and generalized equations. Moreover, we will develop the group theoretical background of the subject which is essential for the Makanin-Razborov diagrams. The foremost observation is that there is a natural one-to-one correspondence between the solutions of the system $\Sigma$ and the homomorphisms from the group

$$G_\Sigma := \langle X | \Sigma \rangle = \langle x_1, \ldots x_n | w_1, w_2 \ldots \rangle$$

(i.e., the group generated by the variables and with the equations as relations) to $G$. This allows us to desribe the set of solutions up to automorphisms of $G_\Sigma$, which is the basic approach that the Makanin-Razborov diagrams take.

As additional aspects, we will finally work out how the situation changes in the case of systems of equations with constants. Moreover, we will (possibly) do an excursion to the closely related topics of the Rips machine and group actions on (simplicial and real) trees.

## References


[M1] G.S. Makanin, *Equations in a free group*, Izv. Akad. Nauk SSSR Ser. Mat. **46** (6), 1982, 1188–1273.

[Ra] A.A. Razborov, *Systems of equations in a free group*, Izv. Akad. Nauk SSSR Ser. Mat. **48** (4), 1984, 779–832.